

Signals and Systems — Tutorial 7

Laplace Transform

Tutorial information

- It is expected that you attempt the prework questions before coming to the tutorial each week and submit your *handwritten* solutions to your tutor within the first 15 minutes of your tutorial.
- Not all questions are equally difficult. Difficult questions are marked with a ★.
- Questions marked **Extension** go beyond the assessable content for this course and are provided to give a deeper theoretical background and understanding.
- You may not be able to complete all questions within the tutorial time. It is *strongly* recommended that you attempt all questions in your own time and ask your tutor if you are stuck.

Pework questions

Consider the transfer function of a system:

$$G(s) = \frac{8s + 4}{s(s + 4)(2s + 4 - 6j)}$$

A: Poles and Zeros

Identify the poles and zeros of the transfer function $G(s)$ and determine whether or not the system is stable, marginally stable or unstable.

B: Initial and final values

Find the initial and final values of the system with transfer function $G(s)$ in response to **(a)** an impulse and **(b)** a step input. If these values don't exist explain why.

C: Fourier Transform

Find the Fourier transform for the signal $g(t)$ with Laplace transform $G(s)$ or if does not exist explain why not.

Tutorial questions

1: Definition of Laplace Transform

- From the definition of the Laplace transform find the transform of the following signals
 - $\cos(3t)H(t)$
 - $e^{-10t}H(t)$
 - $(j + e^{jt})H(t)$
 - $j^t H(-t) + \delta(t - \pi)$
- Find the Fourier transform of the above signals. If a signal does not have a Fourier transform explain why not.

2: Laplace Transforms

Determine the Laplace transform of the following signals

- $e^{-10t} \cos(3t)H(t)$
- $e^{-10t} \cos(3t - 1)H(t)$
- $(2 - 2e^{-4t})H(t)$
- $(t - 1 + e^{-10t} \cos(4t - \pi/3))H(t)$

3: Laplace Transform Properties

A signal $y(t)$, $t \geq 0$, has the Laplace transform

$$Y(s) = \frac{s + 1}{s^2 + 5s + 7}$$

Determine the Laplace transform $V(s)$ of the following signals:

- $v(t) = y(3t - 4)H(3t - 4)$
- $v(t) = ty(t)$
- $v(t) = \frac{d^2 y(t)}{dt^2}$
- $v(t) = (y * H)(t)$
- $v(t) = y(t) \sin(2t)$
- $v(t) = e^{-3t}y(t)$

4: Final and Initial Value Theorem

Determine the final values and initial values of each of the signals with the following Laplace transforms. If either of these do not exist, state why not.

$$Y_1(s) = \frac{4}{s^2 + s}, \quad Y_2(s) = \frac{3s + 4}{s^2 + s} \quad \text{and} \quad Y_3(s) = \frac{4}{s^2 - s}$$

5: Differential Equations and Impulse Response

1. Solve the following differential equations

(a) $\dot{y} + 4y = 3u$ where $u(t) = \sin(2t)$ and $y(0) = 1$

(b) $\ddot{y} + 4\dot{y} + 20y = 2\dot{u} - u$ where $u(t) = H(t)$

2. Given the impulse response $h(t) = (5e^{-3t} + e^{-t})H(t)$ of the LTI system $y(t) = f(u(t))$. Find the transfer function and differential equation of the system.

6: Inverse Laplace Transform

1. Find and determine the stability of the time-domain signals $x_1(t)$, $x_2(t)$ and $x_3(t)$ with Laplace transforms

$$X_1(s) = \frac{10(s+1)}{s^2 + 4s + 3}, \quad X_2(s) = \frac{2s + 100}{(s+1)(s+8)(s+10)} \text{ and } X_3(s) = \frac{10(s+1) - 2s}{s^2 - 4s + 3}$$

- *2. Find and determine the region of convergence of the time-domain signal $x_4(t)$ with Laplace transform

$$X_4(s) = \frac{se^s}{(s-1)(s+1)}$$

7: Transfer function for Series Connections

1. Show that the circuit in Figure 1 with input $u(t) = i(t)$ and output $y(t)$ satisfies

$$RC\dot{u}(t) + u(t) = C\dot{y}(t)$$

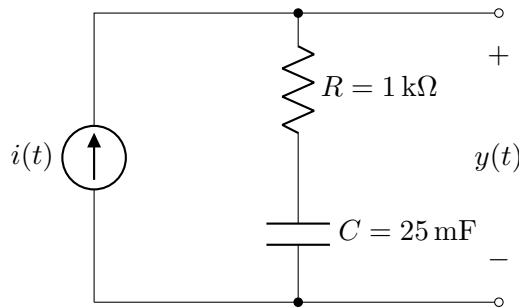


Figure 1: RC-series circuit

2. Find the transfer function $G(s)$ from $u(t)$ to $y(t)$ for this circuit and show that

$$G(s) = G_R(s) + G_C(s)$$

where $G_R(s)$ and $G_C(s)$ are the transfer functions for each of the circuits in Figure 2 with inputs being the current sources and the outputs being the voltages across the resistor and capacitor respectively.



Figure 2: Simple series circuits

8: Convolution and the Laplace transform

Consider the circuit (two first order filter stages connected in cascade by a unity-gain buffer) in Figure 3.

1. Find the transfer functions

$$G_A(s) = \frac{V_A(s)}{U_A(s)} \text{ and } G_B(s) = \frac{V_B(s)}{U_B(s)}$$

2. Show that the impulse response of the first order stage, stage A, is

$$h_A(t) = \delta(t) - \left(\frac{R_A}{L_A} \right) e^{-R_A t / L_A} H(t) \quad (1)$$

and the impulse response of the second first order stage, stage B, is

$$h_B(t) = \left(\frac{1}{R_B C_B} \right) e^{-\frac{t}{R_B C_B}} H(t) \quad (2)$$

i.e. show the voltage across the inductor L_A is $v_A(t) = h_A(t)$ when $u_A(t) = \delta(t)$ and the voltage across the capacitor C_B is $v_B(t) = h_B(t)$ when $u_B(t) = \delta(t)$.

3. The output of a ‘unity-gain buffer’ is the voltage at the non-inverting input terminal i.e. $u_B(t) = v_+(t)$. As stage B is an LTI system we know that $v_B = (h_B * h_A)(t)$. Find $v_B(t)$.
4. Write down the transfer function from the input $u_A(t)$ to the output $v_B(t)$.

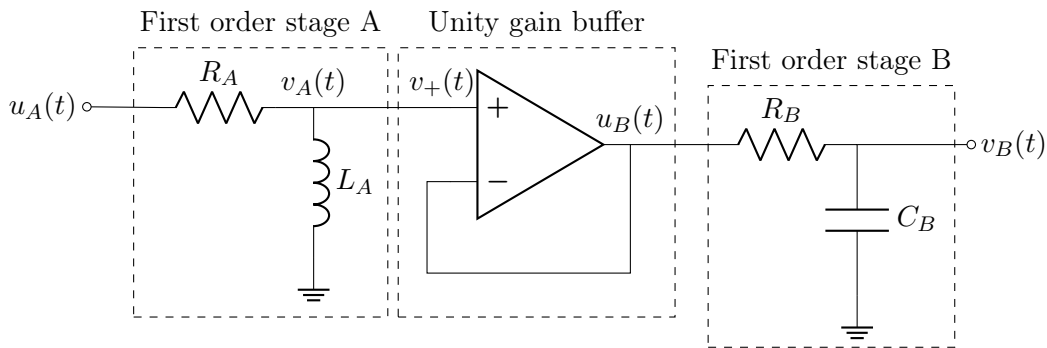


Figure 3: Two first order filter stages (implemented as RL and RC circuits) connected in cascade by a unity-gain voltage buffer i.e. a second order band-pass/stop filter