Week 6 — Lecture notes

Frequency response

The figures below are snapshots of interactive widgets. The live versions, where you can move the sliders yourself, are at tchaffey.com/elec2302.

Lecture outline

0. Recap of Fourier transform
1. Response to arbitrary inputs.
2. Impulse response & convolution with exponentials.
3. Finding the impulse response
4. Interconnection
5. Plotting the impulse response
6. First order systems
7. Second order systems & resonance
8. Periodic inputs.
9. Cascades & string instability.

Recap: the Fourier transform

Expresses a signal as an integral of sinusoidal components.

$$U(j\omega) = \int_{-\infty}^{\infty} u(t)\, e^{-j\omega t}\, \mathrm{d}t$$

(The Fourier transform, generalises Fourier series coefficients).

$$u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(j\omega)\, e^{j\omega t}\, \mathrm{d}\omega$$

(The inverse Fourier transform, generalises the Fourier series).

Transform pairs.

$u(t)$	$U(j\omega)$
$A\, p_\tau(t)$	$A\tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$
$\delta(t)$	1
$e^{-at}H(t)$	$\frac{1}{a+j\omega}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$

Two most important properties:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) \rightsquigarrow j\omega\, U(j\omega) \qquad (1)$$

$$u * y \rightsquigarrow U(j\omega)\,Y(j\omega) \qquad (2)$$

Why are these useful? (1) — find $h(t)$.

(2) — calculate output to any input once we have $h(t)$.

Finding the impulse response

Let's see how. RLC circuit

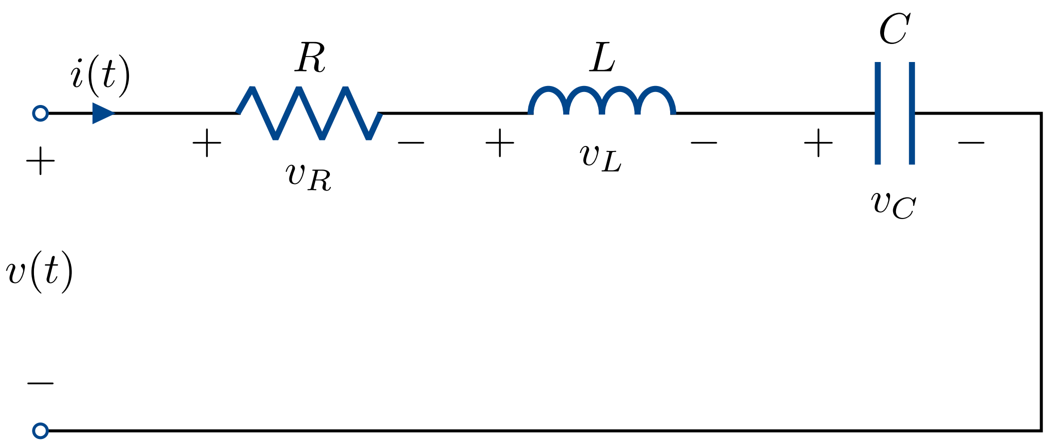

Figure 1: Series RLC circuit.

$$g(t) = 6.25\,e^{-3t}\sin(4t)\,H(t).$$

v input, v_C output.

KVL: $v_R + v_L + v_C = v$.

$$\begin{aligned}
i &= \frac{\mathrm{d}}{\mathrm{d}t} C v_C \\
v_L &= \frac{\mathrm{d}}{\mathrm{d}t} L i \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t^2} L C v_C \\
v_R &= R i \\
&= RC \frac{\mathrm{d}}{\mathrm{d}t} v_C.
\end{aligned}$$

All together: $LC\frac{\mathrm{d}^2}{\mathrm{d}t^2}v_C + RC\frac{\mathrm{d}}{\mathrm{d}t}v_C + v_C = v.$

Now Fourier transform both sides:

$$LC(j\omega)^2 V_C(j\omega) + RC\,j\omega\,V_C(j\omega) + V_C(j\omega) = V(j\omega)$$

If $u(t) = \delta(t)$, $V(j\omega) = 1$.

$$G(j\omega) = V_C(j\omega) = \frac{1}{(j\omega)^2 LC + j\omega RC + 1}$$

Check: same as the end of last lecture!

This is the Fourier transform of the impulse response, which is called the transfer function. Often we'll use $G(j\omega)$ as notation.

Now, given any other input $u(t) \rightsquigarrow U(j\omega)$, we know the output is $y(t) = (u * h)(t) \rightsquigarrow Y(j\omega) = U(j\omega)G(j\omega)$.

Response to arbitrary inputs

Let's see an example. With the values from last week,

$$G(j\omega) = \frac{25}{(j\omega)^2 + 6j\omega + 25}.$$

We also have the spectrum of the Pulse:

$$p_\tau(t) \;\rightsquigarrow\; \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right).$$

Let's shift the pulse so it starts at $t = 0$, not $t = -\dfrac{\tau}{2}$.

$$u(t - t_0) \;\rightsquigarrow\; e^{-j\omega t_0}\,U(j\omega),$$

So

$$p_\tau\left(t - \frac{\tau}{2}\right) \;\rightsquigarrow\; e^{-j\omega\frac{\tau}{2}}\,\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right).$$

If we plug this into the circuit as the supply voltage $v(t)$, the output has Fourier Transform

$$Y(j\omega) = \frac{25}{(j\omega)^2 + 6j\omega + 25}\,e^{-j\omega\frac{\tau}{2}}\,\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right).$$

To get the time domain output, inverse Fourier transform.

Painful by hand, but easy on a computer.

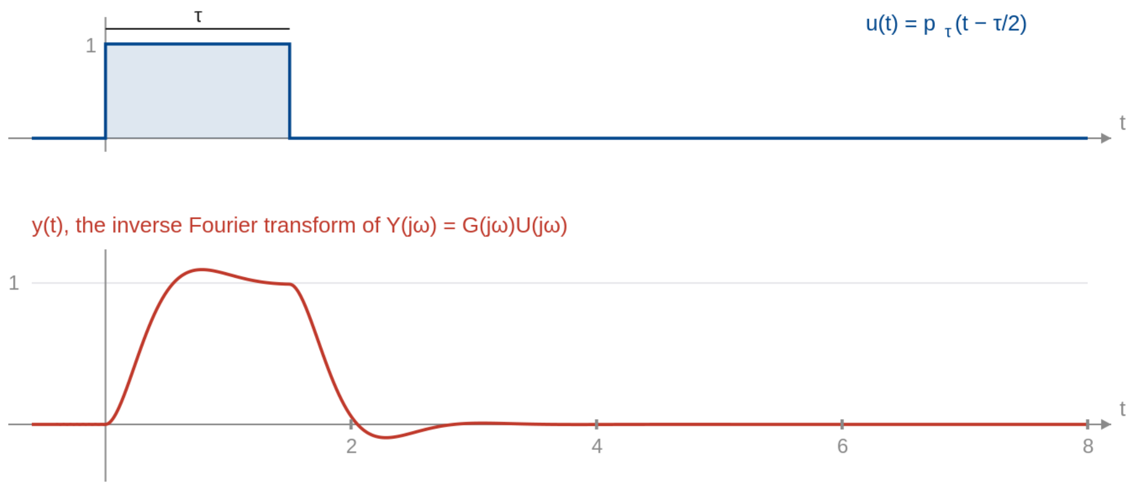

Figure 2: The shifted pulse $u(t) = p_\tau(t - \frac{\tau}{2})$ (top) and the output $y(t)$ of $G(j\omega) = \dfrac{25}{(j\omega)^2 + 6j\omega + 25}$ (bottom), obtained by inverse Fourier transform. Shown at $\tau = 1.5$ s, where the peak output is 1.095 at $t = 0.78$ s.

Convolution with exponentials

What happens to a sinusoidal input $v(t) = e^{j\omega_0 t}$?

In the time domain:

$$\begin{aligned} y = h * v &= \int_{-\infty}^{\infty} v(t-\tau)\, h(\tau)\, \mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} e^{j\omega_0 t}\, e^{-j\omega_0 \tau}\, h(\tau)\, \mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} e^{-j\omega_0 \tau}\, h(\tau)\, \mathrm{d}\tau\, e^{j\omega_0 t} \\ &= G(j\omega_0)\, e^{j\omega_0 t}. \end{aligned}$$

Transfer function scales each frequency!

Interconnections and transfer functions

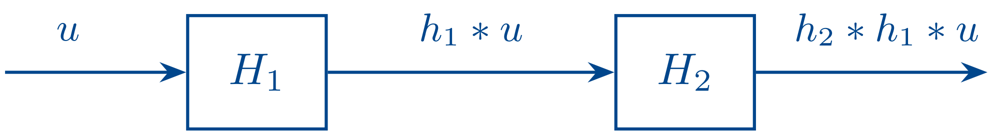

Figure 3: Two systems in cascade.

In the frequency domain, this is just a product: $Y(j\omega) = H_2(j\omega)H_1(j\omega)U(j\omega)$.

What about feedback?

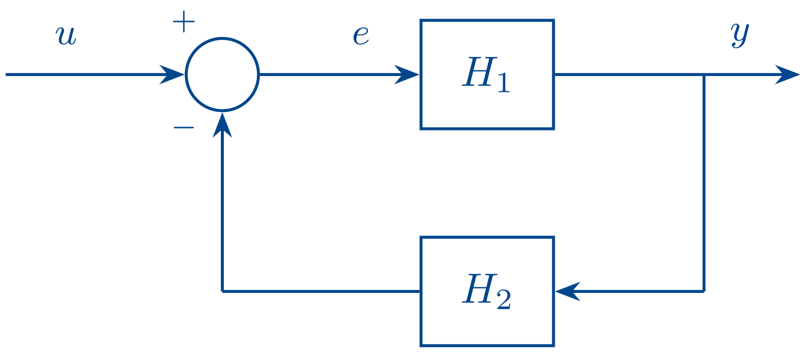

Figure 4: Negative feedback around H_1, with H_2 in the return path.

$$\begin{aligned} E(j\omega) &= U(j\omega) - H_2(j\omega)Y(j\omega) \\ &= U(j\omega) - H_2(j\omega)H_1(j\omega)E(j\omega) \end{aligned}$$

$$E(j\omega) = \frac{1}{1 + H_2(j\omega)H_1(j\omega)}\, U(j\omega).$$

$$Y(j\omega) = H_1(j\omega)E(j\omega) = \frac{H_1(j\omega)}{1 + H_2(j\omega)H_1(j\omega)}\, U(j\omega).$$

Example: good question from last week's lecture.

RC circuit:

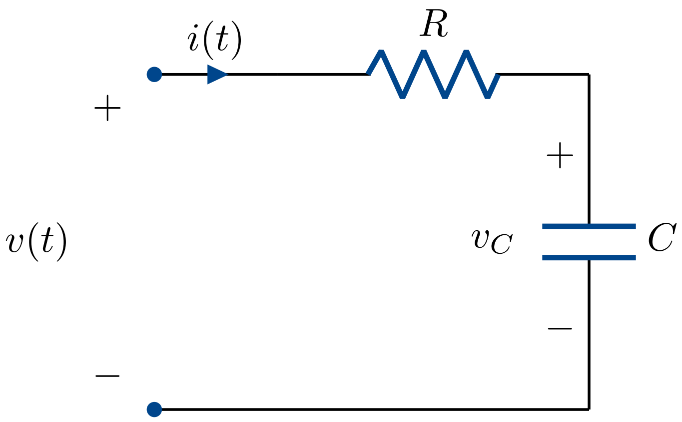

Figure 5: Series RC circuit driven by $v(t)$.

$$v_R = Ri, \qquad V_R(j\omega) = R\,I(j\omega).$$

$$i = \frac{\mathrm{d}}{\mathrm{d}t} C v_C, \qquad I(j\omega) = j\omega C\, V_C(j\omega),$$

$$V_C(j\omega) = \frac{1}{j\omega C}\, I(j\omega).$$

KVL: $v = v_C + v_R$

$$\begin{aligned} V(j\omega) &= V_C(j\omega) + V_R(j\omega) \qquad \text{(Fourier transform is linear)}. \\ &= \left(R + \frac{1}{j\omega C}\right) I(j\omega). \end{aligned}$$

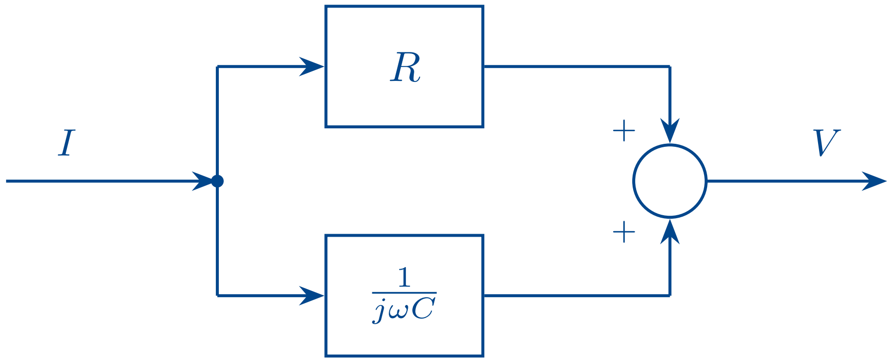

Figure 6: The same relation as a block diagram: I into two parallel branches, summed to give V.

What about I to V?

$$I(j\omega) = \frac{1}{R + \frac{1}{j\omega C}} V(j\omega)$$

$$= \frac{j\omega C}{j\omega RC + 1} V(j\omega).$$

$$\frac{\frac{1}{j\omega C}}{1 + \frac{R}{j\omega C}}$$

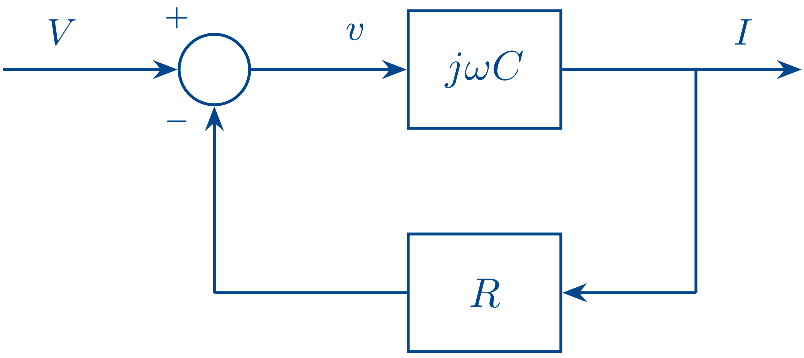

Figure 7: The same relation as feedback: $j\omega C$ in the forward path, R in the return path.

feedback of $j\omega C$ (capacitor maps voltage to current). with R (resistor maps current to voltage).

Decibels

Power & amplitude.

Power gain is measured in power decibels:

$$\alpha^2 \ (\mathrm{V}^2) \ \rightsquigarrow \ 10 \log_{10} \alpha^2 \quad (\mathrm{dB}).$$

Amplitude gain is measured in amplitude decibels:

$$\alpha \ (\mathrm{V}) \ \rightsquigarrow \ 20 \log_{10} \alpha \quad (\mathrm{dB}).$$

Why the factor of 2? $10 \log_{10} \alpha^2 = 20 \log_{10} \alpha$.

Makes power and amplitude consistent.

Plotting the impulse response

There are a few ways to plot the impulse response.

The most common is the Bode Diagram, which plots the argument and amplitude of $G(j\omega)$ on two separate plots.

Amplitude is measured in decibels, $20 \log_{10} |G(j\omega)|$.

Frequency is plotted on a log scale.

For example,

$$\frac{25}{(j\omega)^2 + 6j\omega + 25}$$

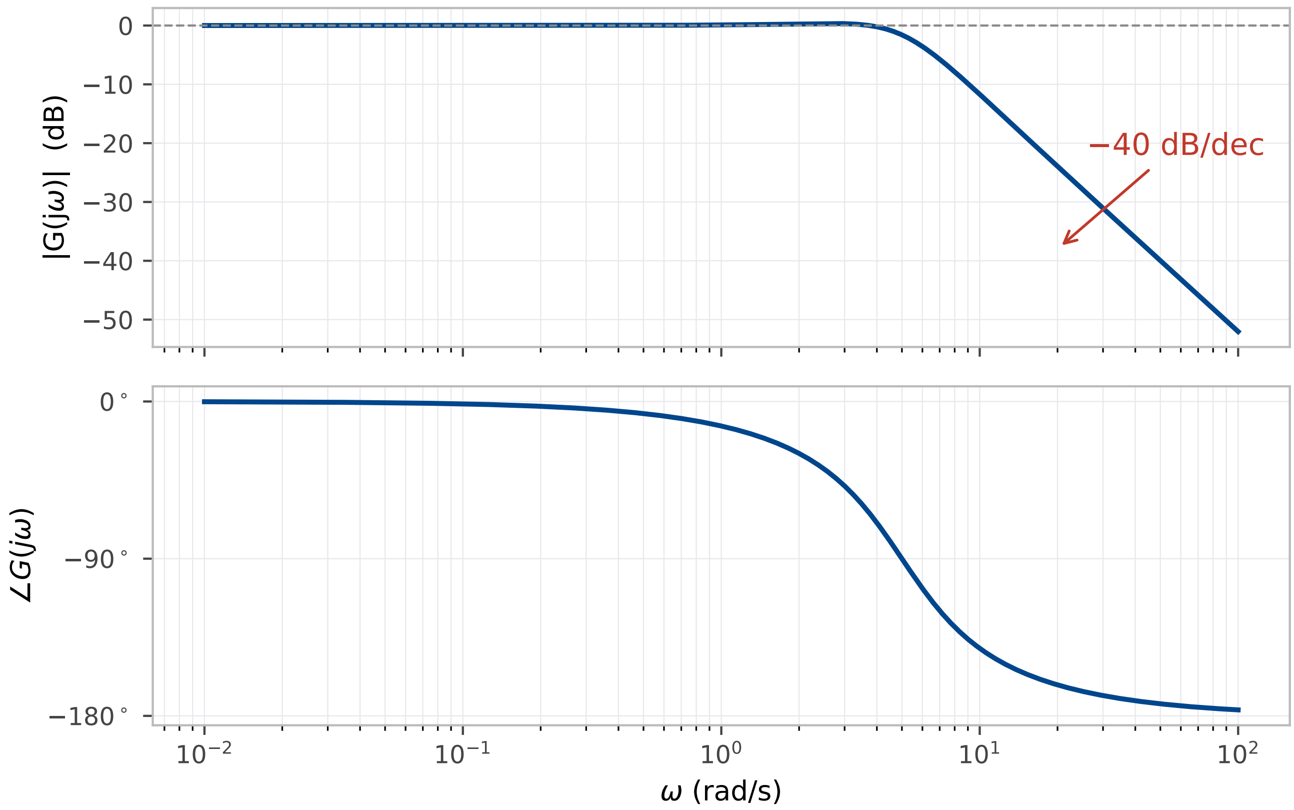

Figure 8: Bode diagram of $G(j\omega) = \frac{25}{(j\omega)^2 + 6j\omega + 25}$, over 0.01 to 100 rad/s.

What does this plot say?

First order systems

(one memory component).

Differential equation:

$$\tau \dot{y} = -y + \beta u$$

$$\tau j\omega Y(j\omega) = -Y(j\omega) + \beta U(j\omega).$$

$$Y(j\omega) = \frac{\beta}{\tau j\omega + 1} U(j\omega).$$

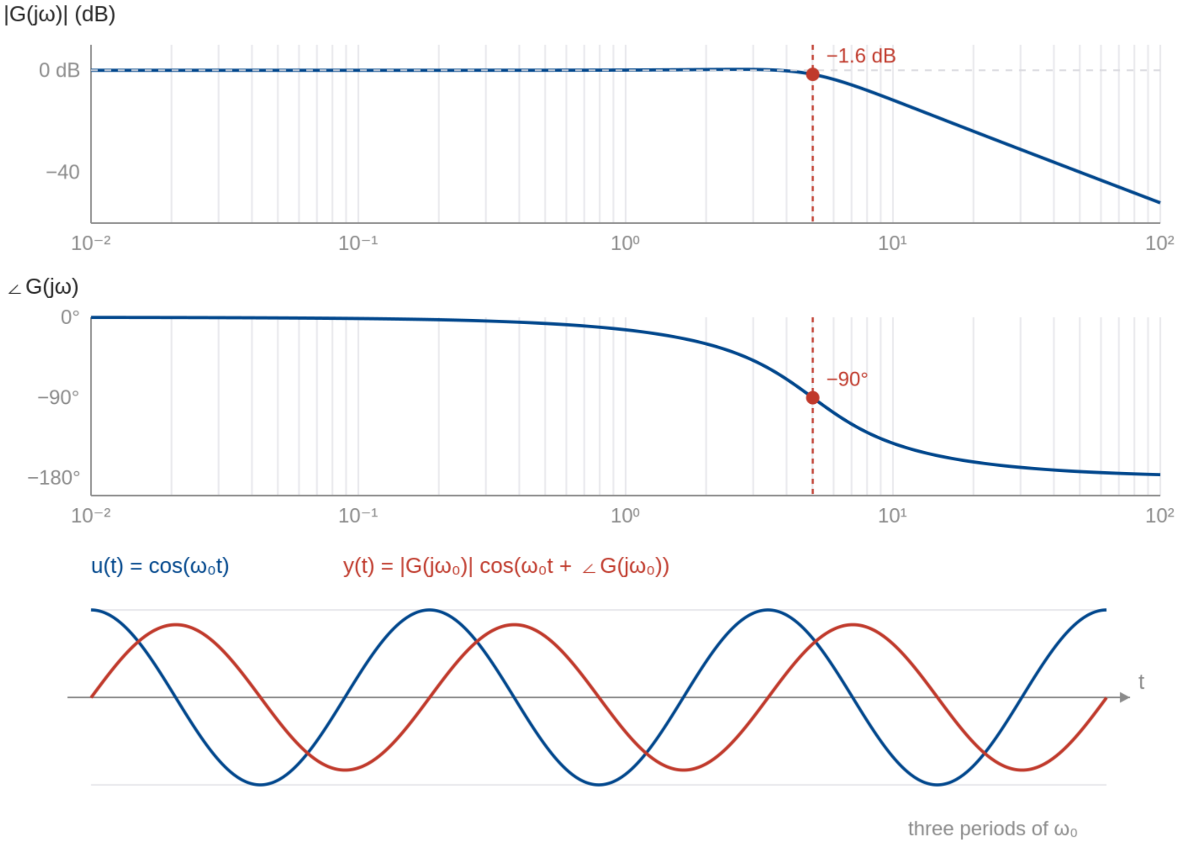

Figure 9: Bode magnitude and phase of $G(j\omega)$ with a marker at ω_0 (top two panels), and the input and output sinusoids at that frequency (bottom). Shown at $\omega_0 = 5.01$ rad/s, where the gain is 0.831 (-1.6 dB) and the phase -90°.

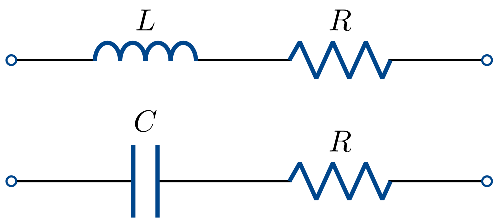

Figure 10: One memory component: an inductor or a capacitor with a resistor.

What does the impulse response look like? (Quiz 1).

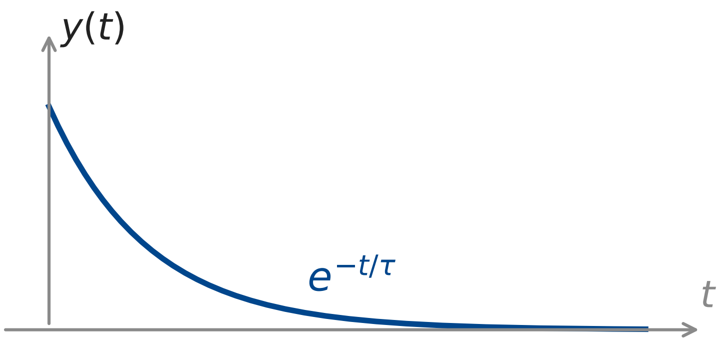

Figure 11: The impulse response of the first order system.

Bode diagram

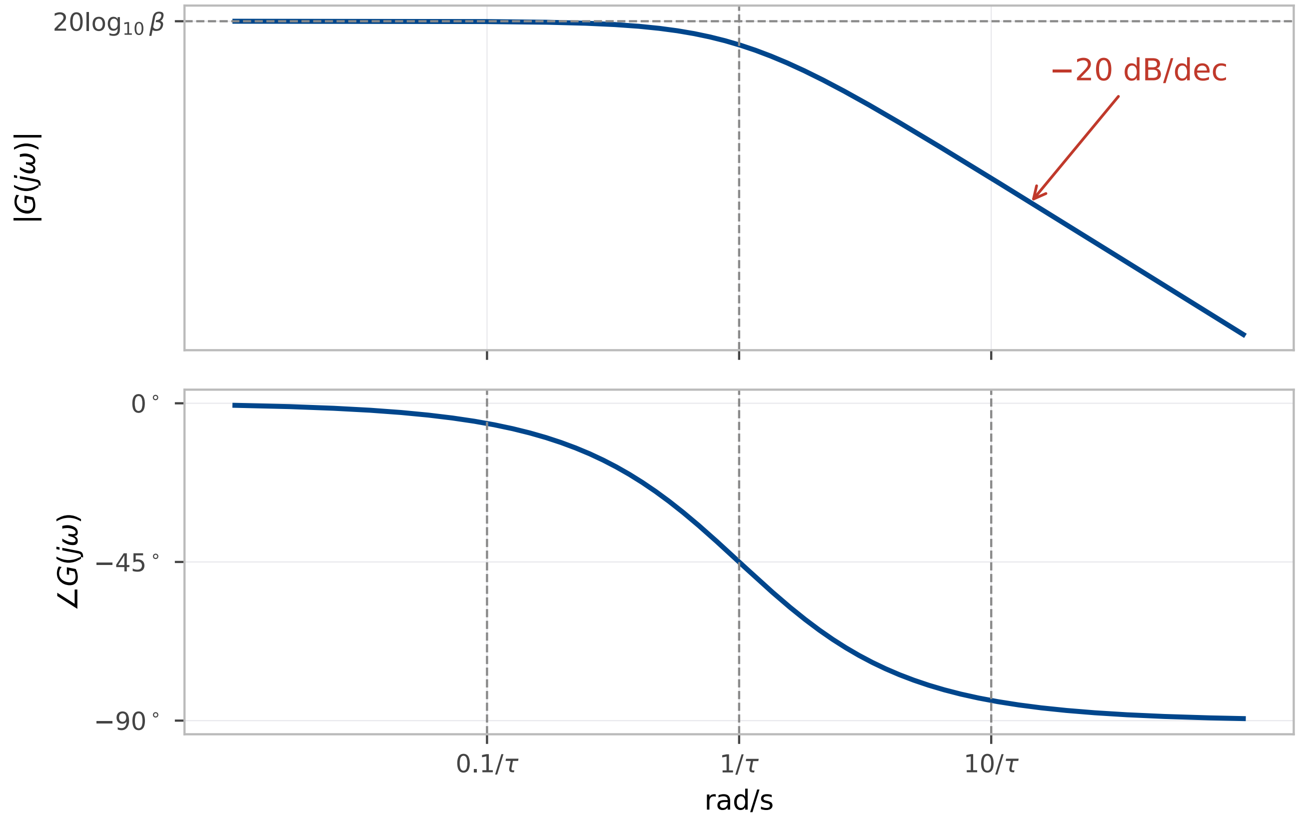

Figure 12: Bode diagram of $\dfrac{\beta}{\tau j\omega + 1}$: the corner is at $1/\tau$, the roll-off is -20 dB/dec, and the phase runs from 0° to -90° over the two decades $0.1/\tau$ to $10/\tau$.

Second order systems

Second order systems have two poles.

We can write a second order system in a standard form:

$$G(j\omega) = \frac{K\omega_n^2}{(j\omega)^2 + 2\omega_n\zeta(j\omega) + \omega_n^2}$$

K, then assumed 1.

ω_n is called the natural frequency.

ζ is called the damping ratio.

The roots of the polynomial $(j\omega)^2 + 2\omega_n\zeta(j\omega) + \omega_n^2$ are the poles:

$$\begin{aligned} j\omega &= \frac{-2\omega_n\zeta \pm \sqrt{4\omega_n^2\zeta^2 - 4\omega_n^2}}{2} \\ &= -\omega_n\left(\zeta \pm \sqrt{\zeta^2 - 1}\right). \end{aligned}$$

Three cases:

$\zeta > 1$: two distinct real poles (overdamped)

$\zeta = 1$: two identical real poles (critically damped)

$\zeta < 1$: complex conjugate poles (underdamped).

Recall from lecture 1: imaginary part gives rotation/oscillation.

Are these the same poles as lecture 1? Yes!

$$G(j\omega) = \frac{A}{(j\omega + p_1)} + \frac{B}{(j\omega + p_2)}.$$

Each pole $\overset{\text{I.F.T.}}{\rightsquigarrow} e^{-p_i t}$.

We can get the Bode diagram of a critically/over damped system by cascading two first order systems:

Figure 13: Two first order systems in cascade.

$$|G(j\omega)| = |G_1(j\omega)G_2(j\omega)|$$

$$\begin{aligned} 20\log_{10}|G_1(j\omega)G_2(j\omega)| &= 20\log_{10}|G_1(j\omega)||G_2(j\omega)| \\ &= 20\log_{10}|G_1(j\omega)| + 20\log_{10}|G_2(j\omega)| \end{aligned}$$

$$\angle G_1(j\omega)G_2(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega)$$

Cascade $\rightsquigarrow$ add magnitudes, add phases.

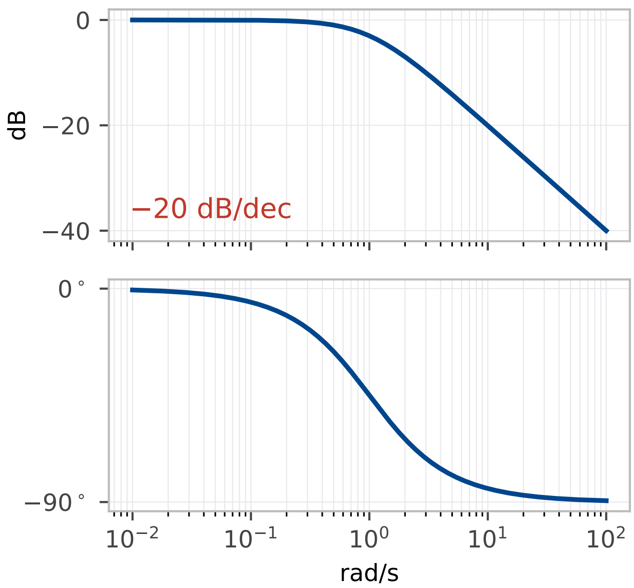

Figure 14: 1st order.

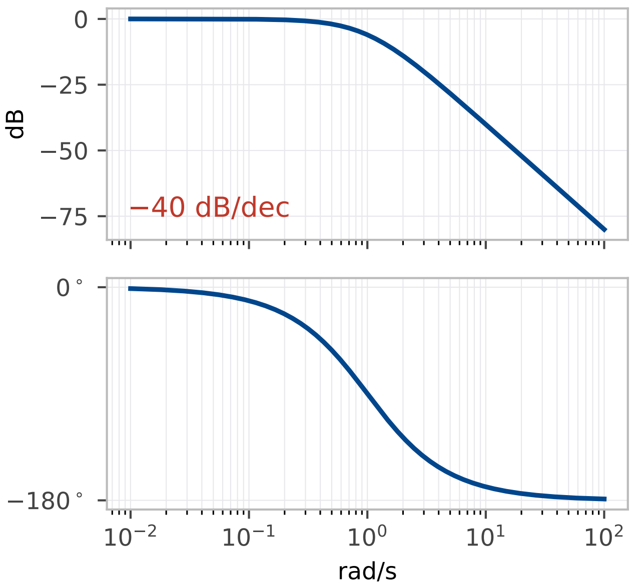

Figure 15: The cascade of two.

What about complex poles?

If the damping is low enough, the system amplifies some frequencies.

$$|G(j\omega)|^2 = \left| \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \right|^2 = \frac{\omega_n^4}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}$$

Let $x = \omega^2$.

$$|G(j\omega)|^2 = \frac{\omega_n^4}{(\omega_n^2 - x)^2 + 4\zeta^2\omega_n^2 x}.$$

Let's minimise the denominator to find the maximum gain:

$$\min_x \left[(\omega_n^2 - x)^2 + 4\zeta^2\omega_n^2 x\right]$$

Diff. w.r.t. x and set to 0:

$$-2(\omega_n^2 - x) + 4\zeta^2\omega_n^2 = 0.$$

$$x = \omega_n^2\left(1 - 2\zeta^2\right).$$

This is positive only when $\zeta < \frac{1}{\sqrt{2}}$.

In this case, $\omega_r = \sqrt{x} = \omega_n\sqrt{1 - 2\zeta^2}$ is the resonant frequency.

The lower the damping ratio, the more the resonant frequency is amplified.

Periodic inputs

We developed the Fourier transform for aperiodic inputs. But if we have a periodic input to a system with aperiodic impulse response, we need to convolve a periodic signal with an aperiodic one. For this, we need the Fourier transform of a periodic signal.

From last week, we know that $e^{j\omega_0 t} \rightsquigarrow 2\pi\delta(\omega - \omega_0)$.

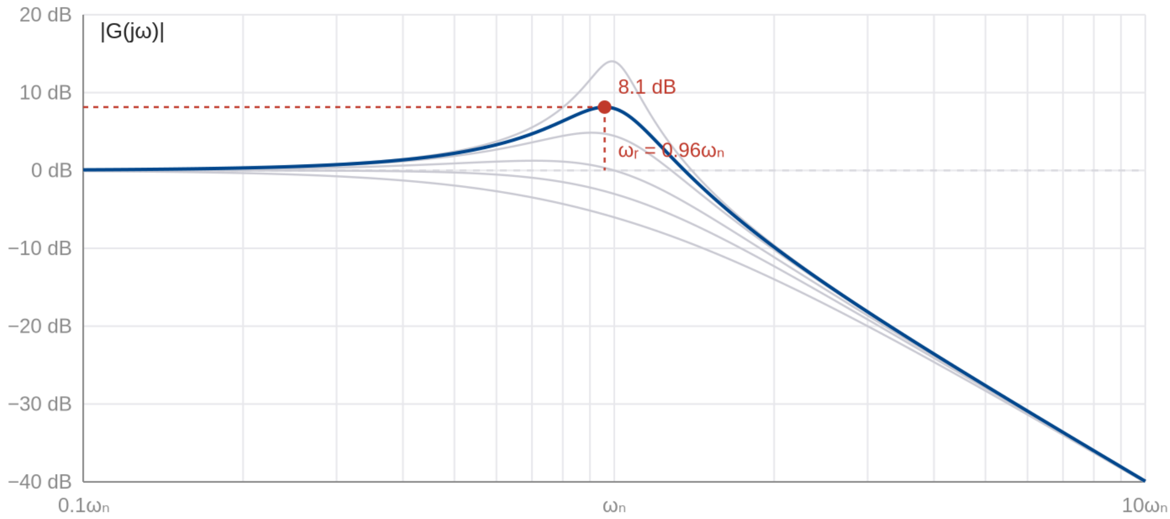

Figure 16: $|G(j\omega)|$ for the standard second order form, over a faint family of damping ratios. Shown at $\zeta = 0.20$, peak gain 2.55 (8.1 dB) at $\omega_r = 0.959\omega_n$.

The Fourier series (week 4 lecture) is

$$u(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{jk\omega_0 t}.$$

Combining, we have

$$U(j\omega) = \sum_{k=-\infty}^{\infty} c_k \, 2\pi \, \delta(\omega - k\omega_0). \qquad (3)$$

Check:

$$\cos(\omega_0 t) = \tfrac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$
$$c_{-1} = \tfrac{1}{2}, \qquad c_1 = \tfrac{1}{2}.$$

In (3): $U(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$.

Example: cascading resonant systems & traffic dynamics

Setup: one car following another.

Figure 17: Car i follows car $i-1$; x_i and x_{i-1} are their positions.

Car is a mass with acceleration governed by throttle & brakes:

$$M\ddot{x}_i = a_i$$
$$\ddot{x}_i = \frac{a_i}{M}.$$

Normalise so $M = 1$.

The driver tries to maintain a constant gap d with the car in front.

If gap is too small/large, driver brakes/accelerates.

$$a_i = k_p \left(x_{i-1} - x_i - d\right).$$

If gap is closing/opening too fast, driver brakes/accelerates more or less:

$$a_i = k_p \left(x_{i-1} - x_i - d\right) + k_d \left(\dot{x}_{i-1} - \dot{x}_i\right)$$

Together:

$$\ddot{x}_i + k_d\dot{x}_i + k_p x_i = k_d\dot{x}_{i-1} + k_p x_{i-1} - k_p d.$$

Let's find the transfer function from x_{i-1} to x_i.

Offset x_{i-1} by d to get rid of the constant term.

$$(j\omega)^2 X_i + k_d\, j\omega X_i + k_p X_i = k_d\, j\omega X_{i-1} + k_p X_{i-1}.$$

$$X_i = \underbrace{\frac{k_d\, j\omega + k_p}{(j\omega)^2 + k_d\, j\omega + k_p}}_{T(j\omega)} X_{i-1}.$$

Now, judging a distance is much easier than judging a rate, so $k_d \ll k_p$.

The damping ratio is $\dfrac{k_d}{2\sqrt{k_p}} \ll 1$: underdamped.

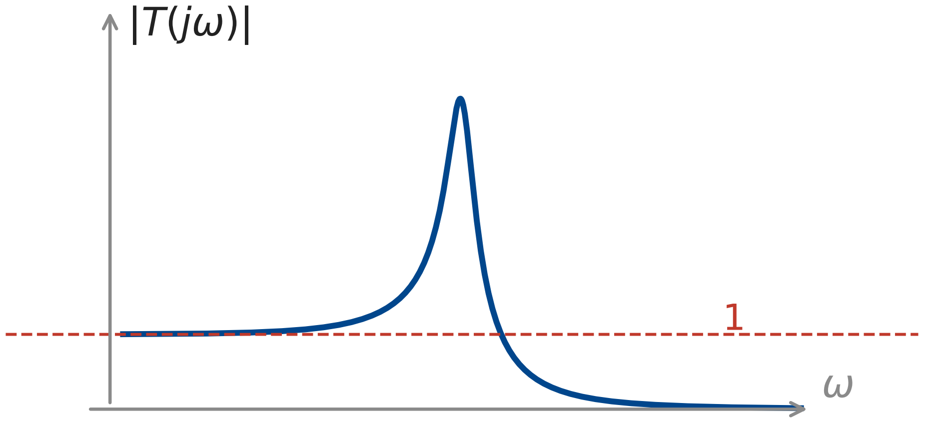

Figure 18: $|T(j\omega)|$ for $k_p = 1$, $k_d = 0.25$: a resonant peak above 1.

What happens if we have many cars following each other? Cascade transfer functions — resonant peaks add. $(T(j\omega))^N$. $|T(j\omega)| > 1$ when $k_p^2 > (k_p - \omega^2)^2$.

Disturbance is amplified: driver at the front tapping their brakes becomes a big oscillation further back in the queue.