Signals and Systems — Tutorial 4

Fourier Transform

Tutorial information

- It is expected that you attempt the prework questions before coming to the tutorial each week and submit your *handwritten* solutions to your tutor within the first 15 minutes of your tutorial.
- Not all questions are equally difficult. Difficult questions are marked with a $\star$.
- Questions marked **Extension** go beyond the assessable content for this course and are provided to give a deeper theoretical background and understanding.
- You may not be able to complete all questions within the tutorial time. It is *strongly* recommended that you attempt all questions in your own time and ask your tutor if you are stuck.

Prework questions

A: Fourier Transform

A radar echo from two targets consists of two impulses, the second delayed by $t_0 > 0$:

$$u(t) = \delta(t) + \delta(t - t_0).$$

1. Sketch $u(t)$, and show from the definition of the Fourier transform that $U(j\omega) = 1 + e^{-j\omega t_0}$.
2. Show that $|U(j\omega)| = 2\,|\cos(\omega t_0/2)|$ and find all ω at which it vanishes. **Hint:** factor out $e^{-j\omega t_0/2}$.
3. Sketch $|U(j\omega)|$ for $-\frac{4\pi}{t_0} \leq \omega \leq \frac{4\pi}{t_0}$.

B: Signals and Systems

For the following examples, write down one possible system that is involved and its input and output signals.

1. your car radio system.
2. a shower with two taps (hot and cold) and a shower head.
3. a person riding a bicycle.
4. an internal combustion engine.

Tutorial questions

1: Fourier Transform

Find the Fourier transform of each of the signals in Figure 1.

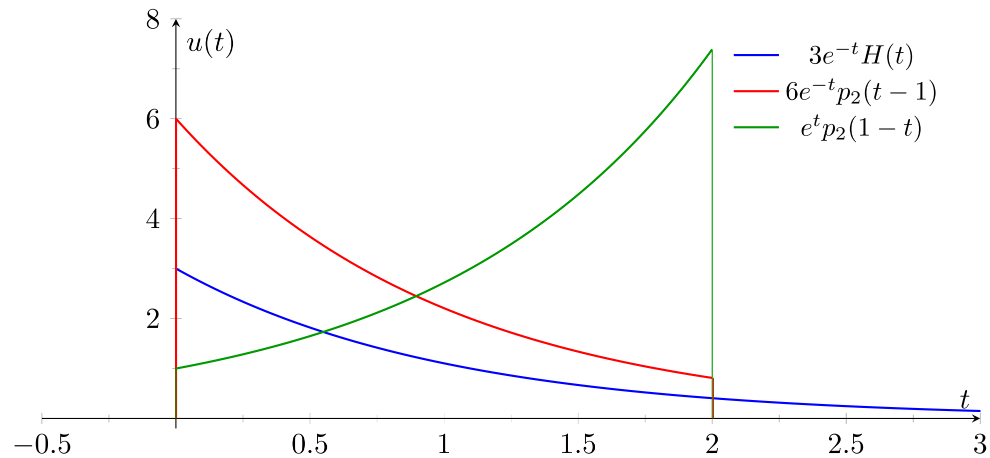

Figure 1: Three signals $u_b(t), u_r(t)$ and $u_g(t)$.

2: Fourier Transform Properties

A continuous-time signal $u(t)$ has Fourier transform

$$U(\omega) = \frac{1}{j\omega + b} \tag{1}$$

where b is a constant. Determine the Fourier transform $Y(\omega)$ of the following signals:

1. $y(t) = u(5t - 4)$
2. $y(t) = u(t)\cos(4t)$
3. $y(t) = \frac{1}{jt-b}$

3: Convolution and the Fourier Transform

Find $(f * g)(t)$ where

1. $f(t) = \text{sinc}(t)$ and $g(t) = \text{sinc}(t)$,
2. $f(t) = \text{sinc}(t)$ and $g(t) = \text{sinc}(2t-2)$ and

$\star$ 3. $f(t) = p_1(t)$ and $g(t) = p_1(t)$.

Hint: The Fourier transform of $p_\tau(t)$ is $\mathcal{F}(p_\tau(t)) = \tau\,\text{sinc}(\omega\tau/(2\pi))$ and $\text{sinc}(\omega) = \frac{\sin(\pi\omega)}{\pi\omega}$

4: Step and Pulse Response of an RC Circuit

Consider the parallel RC circuit from Question 2 and 3 in Tutorial 2 where the input $u(t) = i(t)$ is the current from the source and the output $y(t) = V_C(t)$ is the voltage across the capacitor. Recall that the input $u(t)$ and the output $y(t)$ satisfied the time-domain equation

$$Ru(t) = y(t) + RC\dot{y}(t) \tag{2}$$

which had impulse response $h(t) = e^{-at}$ and that to find the response $y(t)$ to the input $u(t) = H(t)$ – the *step-response* – we had to take the convolution $y(t) = (h * H)(t) = (1 - e^{-at})H(t)$.

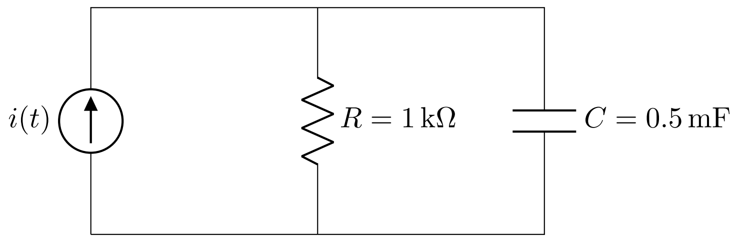

Figure 2: Parallel RC circuit

1. Show that the Fourier transform $Y(j\omega) = \mathcal{F}(y(t))$ satisfies

$$Y(j\omega) = \left(\frac{R}{1 + j\omega RC}\right) U(j\omega) = G(j\omega)U(j\omega) \tag{3}$$

2. **[Extension]** How does this compare to the equivalent impedance of the circuit Z_{eq}?

3. Let $u(t) = \delta(t)$. Use the inverse Fourier transform and (3) to find the impulse response of (2).

$\star$ 4. Now suppose $u_1(t) = H(t)$ and $u_2(t) = p_1(t)$. Use the inverse Fourier transform and (3) to find the step response $y_1(t)$ and 'pulse response' $y_2(t)$ of (3) to the inputs $u_1(t)$ and $u_2(t)$.

5: Radar Resolution and Bandwidth

The Kurnell weather radar transmits the pulse train

$$u(t) = A \sum_{m\in\mathbb{Z}} p_\tau(t - mT), \qquad A = 5\,\mathrm{kV}, \quad \tau = 1\,\mu\mathrm{s}, \quad T = 1\,\mathrm{ms}.$$

1. Find the Fourier transform of a single pulse $Ap_\tau(t)$. Locate its zeros, and hence show that the pulse occupies a bandwidth of roughly $B \approx 1/\tau$. Evaluate B for the radar above.

2. Two targets at ranges r_1 and r_2 return echoes delayed by $t_d = 2r/c$, where $c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$. Show that the echoes overlap when $r_2 - r_1 < c\tau/2$, and evaluate this range resolution Δr for the radar above.

3. Eliminate τ from your answers to Parts 1 and 2 to show that

$$\Delta r \cdot B \approx \frac{c}{2}.$$

   Find the pulse width and bandwidth needed to resolve two aircraft 30 m apart.

4. The designer now doubles the period T, leaving τ unchanged. What happens to the spacing of the spectral lines, to the envelope, and to Δr and B?

5. **[Extension]** The two aircraft of Part 3 return the echo $y = h * u$, where $h(t) = \delta(t) + \delta(t - t_0)$ is the two-impulse echo from Question A in the prework. Write down $Y(j\omega)$ and mark on a sketch the frequencies at which the echo carries no information about the targets.