Signals and Systems — Tutorial 4

Fourier Transform

[Tutor version — includes solutions]

Tutorial information

- It is expected that you attempt the prework questions before coming to the tutorial each week and submit your *handwritten* solutions to your tutor within the first 15 minutes of your tutorial.
- Not all questions are equally difficult. Difficult questions are marked with a $\star$.
- Questions marked **Extension** go beyond the assessable content for this course and are provided to give a deeper theoretical background and understanding.
- You may not be able to complete all questions within the tutorial time. It is *strongly* recommended that you attempt all questions in your own time and ask your tutor if you are stuck.

Prework questions

A: Fourier Transform

A radar echo from two targets consists of two impulses, the second delayed by $t_0 > 0$:

$$u(t) = \delta(t) + \delta(t - t_0).$$

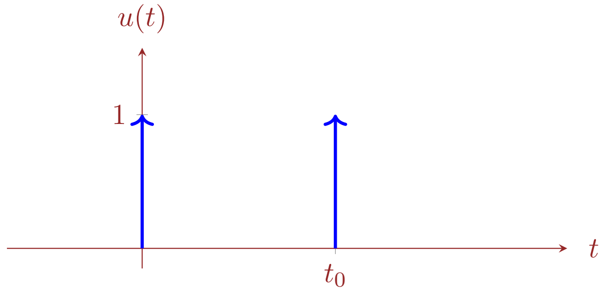
1. Sketch $u(t)$, and show from the definition of the Fourier transform that $U(j\omega) = 1 + e^{-j\omega t_0}$.
2. Show that $|U(j\omega)| = 2\,|\cos(\omega t_0/2)|$ and find all ω at which it vanishes. **Hint:** factor out $e^{-j\omega t_0/2}$.
3. Sketch $|U(j\omega)|$ for $-\frac{4\pi}{t_0} \leq \omega \leq \frac{4\pi}{t_0}$.

B: Signals and Systems

For the following examples, write down one possible system that is involved and its input and output signals.

1. your car radio system.
2. a shower with two taps (hot and cold) and a shower head.
3. a person riding a bicycle.
4. an internal combustion engine.

A **1.** Two unit impulses, one at $t = 0$ and one at $t = t_0$.



From the definition of the Fourier transform and the sifting property,

$$U(j\omega) = \int_{-\infty}^{\infty} (\delta(t) + \delta(t - t_0))\, e^{-j\omega t}\, \mathrm{d}t = e^{-j\omega \cdot 0} + e^{-j\omega t_0} = 1 + e^{-j\omega t_0}.$$

2. Factoring out $e^{-j\omega t_0/2}$,

$$1 + e^{-j\omega t_0} = e^{-j\omega t_0/2}\left(e^{j\omega t_0/2} + e^{-j\omega t_0/2}\right) = 2e^{-j\omega t_0/2}\cos\left(\frac{\omega t_0}{2}\right).$$

The factor $e^{-j\omega t_0/2}$ has modulus 1, so

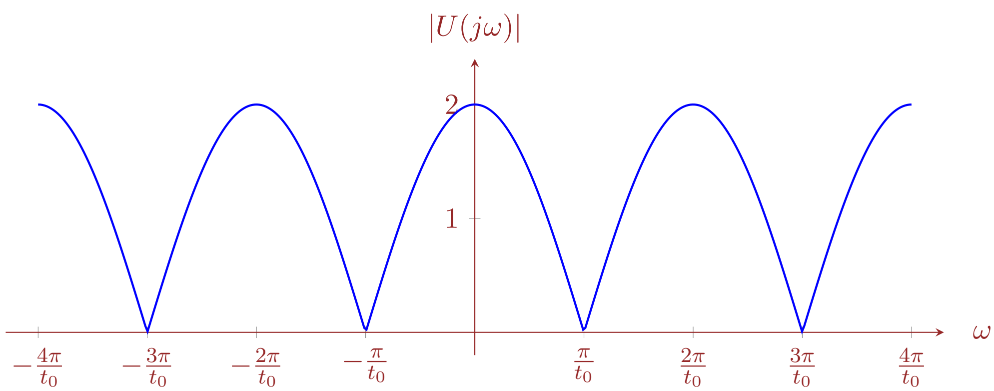
$$|U(j\omega)| = 2\left|\cos\left(\frac{\omega t_0}{2}\right)\right|.$$

This vanishes exactly when $\cos(\omega t_0/2) = 0$, that is when

$$\frac{\omega t_0}{2} = \frac{\pi}{2} + k\pi \iff \omega = \frac{(2k+1)\pi}{t_0}, \qquad k \in \mathbb{Z},$$

so the zeros are at the odd multiples of π/t_0, spaced $2\pi/t_0$ apart.

3. The modulus is a rectified cosine with height 2 at $\omega = 0, \pm 2\pi/t_0, \pm 4\pi/t_0$ and zeros at $\omega = \pm\pi/t_0, \pm 3\pi/t_0$. Taking $t_0 = 1$ gives the following plot:



Tutorial questions

1: Fourier Transform

Find the Fourier transform of each of the signals in Figure 1.

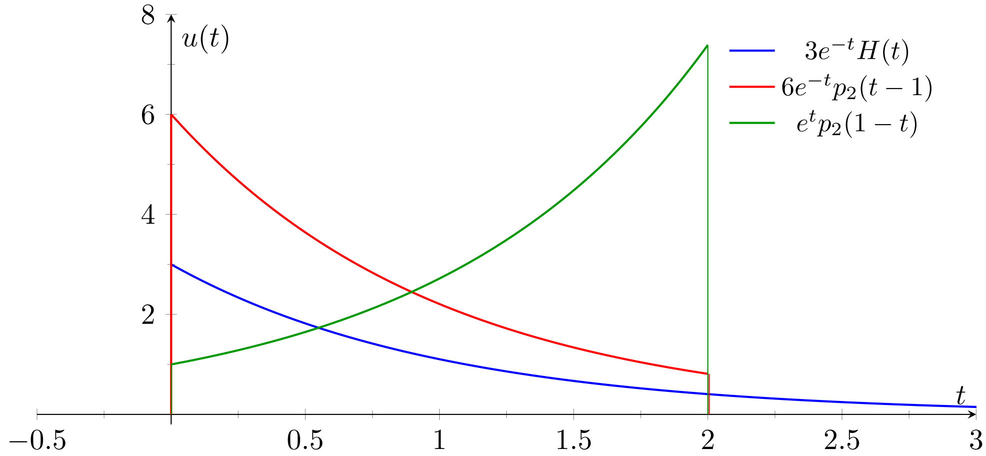


Figure 1: Three signals $u_b(t), u_r(t)$ and $u_g(t)$.

1. u_b From the definition

$$\begin{aligned}\mathcal{F}(3e^{-t}H(t)) &= \int_{-\infty}^{\infty} 3e^{-t}H(t)e^{-j\omega t}\,dt \\ &= 3\int_0^{\infty} e^{-t(1+j\omega)}\,dt \\ &= 3\left.\frac{e^{-t(1+j\omega)}}{-1-j\omega}\right|_0^{\infty} \\ &= \frac{3}{1+j\omega}.\end{aligned}$$

2. u_r Again from the definition

$$\begin{aligned}\mathcal{F}(6e^{-t}p_2(t-1)) &= \int_{-\infty}^{\infty} 6e^{-t}p_2(t-1)e^{-j\omega t}\,dt \\ &= 6\int_0^2 e^{-t(1+j\omega)}\,dt \\ &= 6\left.\frac{e^{-t(1+j\omega)}}{-1-j\omega}\right|_0^2 \\ &= 6\left(\frac{e^{-2(1+j\omega)}}{-1-j\omega}+\frac{1}{1+j\omega}\right) \\ &= 6\left(\frac{1-e^{-2(1+j\omega)}}{1+j\omega}\right)\end{aligned}$$

Alternatively: As $p_2(t-1) = H(t) - H(t-2)$ we can use linearity and the result for $u_b(t)$, together with the time shift. Writing $6e^{-t}H(t-2) = 6e^{-2}e^{-(t-2)}H(t-2)$,

$$\mathcal{F}\left(6e^{-t}p_2(t-1)\right) = \frac{6}{1+j\omega} - \frac{6e^{-2}e^{-2j\omega}}{1+j\omega} = \frac{6\left(1-e^{-2(1+j\omega)}\right)}{1+j\omega}.$$

3. u_g Use *time-reversal* i.e. $u(-t) \leftrightarrow U(-j\omega)$, linearity and Fourier transform for $u_r(t)$. We have

$$u_g(t) = \frac{u_r(-t)}{6} \leftrightarrow U_g(j\omega) = \frac{U_r(-j\omega)}{6} = \frac{1-e^{-2(1-j\omega)}}{1-j\omega}$$

2: Fourier Transform Properties

A continuous-time signal $u(t)$ has Fourier transform

$$U(j\omega) = \frac{1}{j\omega+b} \tag{1}$$

where b is a constant. Determine the Fourier transform $Y(j\omega)$ of the following signals:

1. $y(t) = u(5t-4)$
2. $y(t) = u(t)\cos(4t)$
3. $y(t) = \frac{1}{jt-b}$

1. This is *time scaling* by 5 and a *time shift* by 4.

Time scaling by a:

$$u(at) \longleftrightarrow \frac{1}{|a|}U\left(\frac{j\omega}{a}\right)$$

Time shift:

$$u(t-c) \longleftrightarrow U(j\omega)\,e^{-j\omega c}$$

Hence

$$u(5t-4) = u\left(5\left(t-\frac{4}{5}\right)\right) \longleftrightarrow \frac{1}{5}U\left(\frac{j\omega}{5}\right)e^{-j\frac{4}{5}\omega} = \frac{e^{-j\frac{4}{5}\omega}}{j\omega+5b}$$

2. Multiplication by a trigonometric function or equivalently *linearity* and multiplication by a complex exponential.

   Linearity for $\alpha \in \mathbb{C}$

   $$u_1(t) + \alpha u_2(t) \leftrightarrow U_1(j\omega) + \alpha U_2(j\omega)$$

   Multiplication by complex exponential

   $$u(t)e^{j\omega_0 t} \leftrightarrow U(j(\omega-\omega_0))$$

   $$y(t) = u(t)\cos(4t) = \frac{1}{2}\left(u(t)e^{4jt} + u(t)e^{-4jt}\right) \leftrightarrow \frac{1}{2}\left(U(j(\omega-4)) + U(j(\omega+4))\right)$$

   So

   $$Y(j\omega) = \frac{1}{2}\left(\frac{1}{j(\omega-4)+b} + \frac{1}{j(\omega+4)+b}\right) = \left(\frac{j\omega+b}{(j\omega)^2+2j\omega b+b^2+16}\right) = \left(\frac{j\omega+b}{(j\omega+b)^2+16}\right)$$

3. This uses **Duality**, which for a pair $u(t) \leftrightarrow U(j\omega)$ reads

   $$U(jt) \longleftrightarrow 2\pi u(-\omega)$$

   The inverse Fourier transform of $U(j\omega) = 1/(j\omega+b)$ is $u(t) = e^{-bt}H(t)$. First, notice that

   $$y(t) = \frac{1}{jt-b} = \frac{-1}{j(-t)+b} = -U(j(-t))$$

   Hence

   $$Y(j\omega) = -2\pi u(\omega) = -2\pi e^{-b\omega}H(\omega) = \begin{cases} -2\pi e^{-b\omega}, & \omega \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

3: Convolution and the Fourier Transform

Find $(f * g)(t)$ where

1. $f(t) = \text{sinc}(t)$ and $g(t) = \text{sinc}(t)$,
2. $f(t) = \text{sinc}(t)$ and $g(t) = \text{sinc}(2t-2)$ and

⋆ 3. $f(t) = p_1(t)$ and $g(t) = p_1(t)$.

Hint: The Fourier transform of $p_\tau(t)$ is $\mathcal{F}(p_\tau(t)) = \tau\,\text{sinc}(\omega\tau/(2\pi))$ and $\text{sinc}(\omega) = \frac{\sin(\pi\omega)}{\pi\omega}$

Throughout we will use that convolution in the time domain is equivalent to multiplication in the frequency domain i.e. $(u * v)(t) \leftrightarrow U(\omega)V(\omega)$.

1.
$$(\mathrm{sinc} * \mathrm{sinc})(t) \leftrightarrow p_{2\pi}(\omega)^2 = p_{2\pi}(\omega) \leftrightarrow \mathrm{sinc}(t)$$

2. Use time-scaling and time-shifting.
$$(f * g)(t) \leftrightarrow p_{2\pi}(\omega)\frac{1}{2}p_{4\pi}(\omega)e^{-j\omega} = \frac{1}{2}p_{2\pi}(\omega)e^{-j\omega} \leftrightarrow \frac{1}{2}\,\mathrm{sinc}(t-1)$$

3.
$$(p_1 * p_1)(t) \leftrightarrow \mathrm{sinc}^2(\omega/(2\pi)) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^2$$

We now need to find the inverse Fourier transform
$$I(t) = \mathcal{F}^{-1}\left(\mathrm{sinc}^2(\omega/(2\pi))\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin(\omega/2)}{\omega/2}\right)^2 e^{j\omega t}\,d\omega = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2(\omega/2)}{\omega^2}\right)e^{j\omega t}\,d\omega$$

Taking the second derivative with respect to t we have
$$\begin{aligned} I''(t) &= -\frac{2}{\pi}\int_{-\infty}^{\infty}\sin^2(\omega/2)e^{j\omega t}\,d\omega \\ &= -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos(\omega)}{2}e^{j\omega t}\,d\omega \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}2(1-\cos(\omega))e^{j\omega t}\,d\omega \\ &= -\mathcal{F}^{-1}(2) + \mathcal{F}^{-1}(2\cos(\omega)) \\ &= -2\delta(t) + (\delta(t-1)+\delta(t+1)) \end{aligned}$$

We now need to integrate twice:
$$I'(t) = \int_{-\infty}^{t} I''(\tau)\,\mathrm{d}\tau = -2H(t) + H(t-1) + H(t+1) = \begin{cases} 0, & t \notin [-1,1) \\ 1, & t \in [-1,0) \\ -1 & t \in [0,1) \end{cases}$$

Integrating again we have
$$I(t) = \int_{-\infty}^{t} I'(\tau)\,\mathrm{d}\tau = (1+t)H(t+1)H(-t) + (1-t)H(t)H(1-t)$$

i.e. a triangular wave with width 2 centred at 0 with height 1. Both constants of integration are zero because I and I' vanish as $t \to -\infty$.

4: Step and Pulse Response of an RC Circuit

Consider the parallel RC circuit from Question 2 and 3 in Tutorial 2 where the input $u(t) = i(t)$ is the current from the source and the output $y(t) = V_C(t)$ is the voltage across the capacitor. Recall that the input $u(t)$ and the output $y(t)$ satisfied the time-domain equation

$$Ru(t) = y(t) + RC\dot{y}(t) \tag{2}$$

which had impulse response $h(t) = \frac{1}{C}e^{-t/(RC)}H(t)$ and that to find the response $y(t)$ to the input $u(t) = H(t)$ – the *step-response* – we had to take the convolution $y(t) = (h * H)(t) = R\left(1 - e^{-t/(RC)}\right)H(t)$.

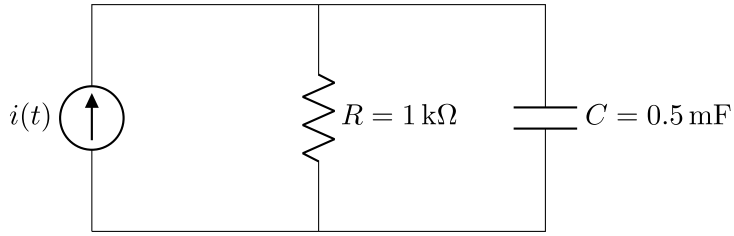


Figure 2: Parallel RC circuit

1. Show that the Fourier transform $Y(j\omega) = \mathcal{F}(y(t))$ satisfies

$$Y(j\omega) = \left(\frac{R}{1 + j\omega RC}\right)U(j\omega) = G(j\omega)U(j\omega) \tag{3}$$

2. **[Extension]** How does this compare to the equivalent impedance of the circuit Z_{eq}?

3. Let $u(t) = \delta(t)$. Use the inverse Fourier transform and (3) to find the impulse response of (2).

⋆ 4. Now suppose $u_1(t) = H(t)$ and $u_2(t) = p_1(t)$. Use the inverse Fourier transform and (3) to find the step response $y_1(t)$ and 'pulse response' $y_2(t)$ of (3) to the inputs $u_1(t)$ and $u_2(t)$.

1. Applying the Fourier transform to (2) we have

$$\mathcal{F}(Ru) = \mathcal{F}(y + RC\dot{y})$$

Using linearity

$$R\mathcal{F}(u) = \mathcal{F}(y) + RC\mathcal{F}(\dot{y})$$

As $\mathcal{F}(\dot{y}) = j\omega\mathcal{F}(y)$ we have

$$RU = Y + j\omega RCY$$

Rearranging gives

$$Y = \frac{R}{1 + j\omega RC}U(j\omega)$$

where $G(j\omega) = \frac{R}{1+j\omega RC}$.

2. As the circuit components are in parallel with the impedances $Z_R = R$ and $Z_C = 1/(j\omega C)$ we have

$$Z_{eq} = \frac{Z_R Z_C}{Z_R + Z_C} = \frac{R/(j\omega C)}{R + \frac{1}{j\omega C}} = \frac{R}{j\omega RC + 1} = G(j\omega)$$

So Fourier Transform of (2) is identically the equivalent impedance i.e. $G(j\omega) = Z_{eq}$ is the Frequency Response Function or Transfer Function of the circuit.

3. Impulse response is given by setting $U(j\omega) = 1$ which means that $Y(j\omega) = G(j\omega)$. Taking the inverse Fourier transform we have

$$h(t) = \mathcal{F}^{-1}\left(G(j\omega)\right) = \frac{1}{C}\mathcal{F}^{-1}\left(\frac{1}{1/(RC) + j\omega}\right) = \left(\frac{1}{C}\right) e^{-\frac{t}{RC}} H(t)$$

The $H(t)$ is very important as seen in the next question.

4. As we know the impulse response $h(t)$ and the response y of a LTI system to any other input u is given by $y = h * u$ we can find $y_1(t)$ by multiplying the Fourier transforms of $h(t)$ and $H(t)$, using $\mathcal{F}\left(H(t)\right) = \pi\delta(\omega) + \frac{1}{j\omega}$, i.e.

$$\begin{aligned}
y_1(t) &= \mathcal{F}^{-1}\left(\left(\frac{R}{1 + j\omega RC}\right)\left(\pi\delta(\omega) + \frac{1}{j\omega}\right)\right) \\
&= \mathcal{F}^{-1}\left(\left(\frac{\pi R}{1 + j\omega RC}\right)\delta(\omega) + \left(\frac{1}{j\omega}\right)\left(\frac{R}{1 + j\omega RC}\right)\right) \\
&= \mathcal{F}^{-1}\left(\pi G(j\omega)\delta(\omega) + \frac{1}{j\omega}G(j\omega)\right)
\end{aligned}$$

As $\delta(\omega) = 0$ for all $\omega \neq 0$ we have

$$y_1(t) = \mathcal{F}^{-1}\left(\pi G(0)\delta(\omega) + \frac{1}{j\omega}G(j\omega)\right)$$

Using the Fourier transform property

$$\int_{-\infty}^{t} g(\tau)\, d\tau \leftrightarrow \frac{1}{j\omega}G(j\omega) + \pi G(0)\delta(\omega)$$

We have

$$\begin{aligned} y_1(t) &= \mathcal{F}^{-1}\left(\pi G(j\omega)\delta(\omega) + \frac{1}{j\omega}G(j\omega)\right) \\ &= \int_{-\infty}^{t} h(\tau)\,\mathrm{d}\tau \\ &= \int_{-\infty}^{t} \frac{e^{-\tau/(RC)}}{C} H(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{C}\int_{0}^{t} e^{-\tau/(RC)}\,d\tau \\ &= R\left(1 - e^{-t/(RC)}\right)H(t) \end{aligned}$$

Alternatively

$$\begin{aligned} y_1(t) &= \mathcal{F}^{-1}\left(\pi G(0)\delta(\omega) + \frac{1}{j\omega}G(j\omega)\right) \\ &= \pi G(0)\mathcal{F}^{-1}\left(\delta(\omega)\right) + R\mathcal{F}^{-1}\left(\frac{1}{(j\omega)\,(1 + j\omega RC)}\right) \\ &= \frac{\pi G(0)}{2\pi} + \mathcal{F}^{-1}\left(\frac{R}{j\omega} - \frac{R^2 C}{1 + j\omega RC}\right) \\ &= \frac{R}{2} + \frac{R}{2}\operatorname{sgn}(t) - Re^{-t/(RC)}H(t) \\ &= R\left(1 - e^{-t/(RC)}\right)H(t) \end{aligned}$$

using $\mathcal{F}^{-1}\left(\frac{1}{j\omega}\right) = \frac{1}{2}\operatorname{sgn}(t)$ and $\mathcal{F}^{-1}\left(\frac{R^2C}{1+j\omega RC}\right) = Re^{-t/(RC)}H(t)$, and then $\frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t) = H(t)$.

'Pulse response' $y_2(t)$ For convenience of notation we apply a time-shift to $p_1(t)$ and consider $p_s(t) = p_1(t - 1/2)$. As $p_1(t - 1/2) = H(t) - H(t-1)$ and the convolution is linear we have

$$y_2(t) = (h * p_s)(t) = (h * H)(t) - (h * H)(t-1)$$

Using the response for $y_1(t)$ we have

$$y_2(t) = R(1 - e^{-t/(RC)})H(t) - R(1 - e^{-(t-1)/RC})H(t-1)$$

i.e.

$$y_2(t) = \begin{cases} 0, & t < 0 \\ R(1 - e^{-t/(RC)}), & t \in [0,1] \\ R(e^{-(t-1)/(RC)} - e^{-t/(RC)}), & t > 1 \end{cases} = \begin{cases} 0, & t < 0 \\ R(1 - e^{-t/(RC)}), & t \in [0,1] \\ Re^{-t/(RC)}\left(e^{1/(RC)} - 1\right) & t > 1 \end{cases}$$

Checking continuity of the solution at $t = 0$ and $t = 1$

$$0 = R(1 - e^{-0/(RC)}) = 0 \text{ and } R(1 - e^{-1/(RC)}) = R\left(e^{-0/(RC)} - e^{-1/(RC)}\right)$$

Finally, since $p_1(t) = p_s(t + 1/2)$, applying $t \mapsto t + 1/2$

$$y_2(t) = \begin{cases} 0, & t < -1/2 \\ R(1 - e^{-(t+1/2)/(RC)}), & t \in [-1/2, 1/2] \\ Re^{-(t-1/2)/(RC)}\left(1 - e^{-1/(RC)}\right) & t > 1/2 \end{cases}$$

5: Radar Resolution and Bandwidth

The Kurnell weather radar transmits the pulse train

$$u(t) = A \sum_{m \in \mathbb{Z}} p_\tau(t - mT), \qquad A = 5\,\mathrm{kV}, \quad \tau = 1\,\mu\mathrm{s}, \quad T = 1\,\mathrm{ms}.$$

1. Find the Fourier transform of a single pulse $Ap_\tau(t)$. Locate its zeros, and hence show that the pulse occupies a bandwidth of roughly $B \approx 1/\tau$. Evaluate B for the radar above.
2. Two targets at ranges r_1 and r_2 return echoes delayed by $t_d = 2r/c$, where $c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$. Show that the echoes overlap when $r_2 - r_1 < c\tau/2$, and evaluate this range resolution Δr for the radar above.
3. Eliminate τ from your answers to Parts 1 and 2 to show that

$$\Delta r \cdot B \approx \frac{c}{2}.$$

   Find the pulse width and bandwidth needed to resolve two aircraft 30 m apart.
4. The designer now doubles the period T, leaving τ unchanged. What happens to the spacing of the spectral lines, to the envelope, and to Δr and B?
5. **[Extension]** The two aircraft of Part 3 return the echo $y = h * u$, where $h(t) = \delta(t) + \delta(t - t_0)$ is the two-impulse echo from Question A in the prework. Write down $Y(j\omega)$ and mark on a sketch the frequencies at which the echo carries no information about the targets.

1. From lectures we know $\mathcal{F}(p_\tau(t)) = \tau\,\mathrm{sinc}\,(\omega\tau/(2\pi))$. Let's derive it here for practice using $\tau = 1$.

$$\mathcal{F}(p_1(t)) = \int_{-\infty}^{\infty} p_1(t)e^{-j\omega t}\,\mathrm{d}t = \int_{-1/2}^{1/2} e^{-j\omega t}\,\mathrm{d}t = \frac{1}{j\omega}\left(e^{j\omega/2} - e^{-j\omega/2}\right)$$

   Using the identity $2j\sin(\omega/2) = e^{j\omega/2} - e^{-j\omega/2}$, we have

$$\frac{1}{j\omega}\left(e^{j\omega/2} - e^{-j\omega/2}\right) = \frac{2j\sin(\omega/2)}{j\omega} = \frac{\sin(\omega/2)}{\omega/2} = \frac{\sin(\pi\omega/(2\pi))}{\pi\omega/(2\pi)} = \mathrm{sinc}(\omega/(2\pi))$$

Using linearity of the Fourier transform we have that

$$\begin{aligned}\mathcal{F}(Ap_\tau(t)) &= A\mathcal{F}(p_\tau(t)) \\ &= A\tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right) \\ &= (5\times 10^3)(10^{-6}) \operatorname{sinc}\left(\frac{10^{-6}\omega}{2\pi}\right) \\ &= 5\times 10^{-3} \operatorname{sinc}\left(\frac{\omega}{2\pi\times 10^6}\right)\end{aligned}$$

To find the zeros of the Fourier transform $\tau\operatorname{sinc}(\omega\tau/(2\pi)) = 0$ we use the definition $\operatorname{sinc}(\omega) = \sin(\pi\omega)/(\pi\omega)$ which is zero if and only if $\sin(\pi\omega) = 0$. Thus

$$\tau\operatorname{sinc}(\omega\tau/(2\pi)) = 0 \iff \frac{2\tau\sin(\omega\tau/2)}{\omega\tau} = 0 \iff \sin(\omega\tau/2) = 0 \iff \omega\tau/2 = k\pi$$

for $k \in \mathbb{Z}\backslash\{0\}$: at $k = 0$ the transform equals τ, which is its peak and not a zero. Thus the zeros occur at $\omega = 2k\pi/\tau$, $k \neq 0$. The bandwidth is the distance from 0 to the first zero of $U(j\omega)$ normalised by 2π, converting rad s^{-1} to Hz, which is $1/\tau$. For the radar above $B = 1/(1\,\mu\text{s}) = 1\,\text{MHz}$.

2. Suppose pulse 1 arrives at time t_1'. As the pulse has width τ this implies if pulse 2 arrives before $t = t_1' + \tau$ then its arrival will overlap with pulse 1 i.e. for the pulses to be separate we require $t_2' > t_1' + \tau$. Using this and that we know $t_1' = 2r_1/c$ and $t_2' = 2r_2/c$. We have $r_2 - r_1 > \tau c/2$ for the pulses to be separate. So the range resolution is

$$\Delta r = \frac{c\tau}{2} = 150\,\text{m}$$

Note the formula $t_d = 2r/c$ is from the formula $v = d/t$ where the velocity is c and the distance is $2r$ as the pulse must travel and then return from the target.

Since p_τ is centred, echo 1 occupies $[t_1' - \tau/2,\, t_1' + \tau/2]$ and echo 2 occupies $[t_2' - \tau/2,\, t_2' + \tau/2]$. These are disjoint exactly when $|t_2' - t_1'| > \tau$, the same condition as above.

3. From above we have

$$\Delta r \cdot B = \left(\frac{c\tau}{2}\right)\left(\frac{1}{\tau}\right) = \frac{c}{2}$$

If two aircraft are 30 m apart we require $\Delta r \leq 30\,\text{m}$ i.e.

$$\tau \leq 2\cdot 30/(3\times 10^8) = 200\,\text{ns}$$

which implies the Bandwidth $B = 1/\tau \geq 5\,\text{MHz}$.

4. Recall from Fourier series the fundamental frequency is $\omega_0 = 2\pi/T$. Therefore if we double $T' = 2T$ we halve the fundamental frequency i.e. $\omega_0' = \omega_0/2$. Each spectral line occurs at an integer multiple of the fundamental $k\omega_0$. So the spacing $\Delta\omega'$ is halved as

$$\Delta\omega' = k\omega_0' - (k-1)\omega_0' = \omega_0' = \omega_0/2$$

or

$$\Delta\omega = k\omega_0 - (k-1)\omega_0 = \omega_0 = 2\omega_0'$$

As we have not changed τ the *envelope* keeps its shape and its nulls at $\omega = 2k\pi/\tau$, and similarly Δr and B have not changed.

5. $Y(j\omega) = H(j\omega)U(j\omega)$, where $H(j\omega) = 1 + e^{-j\omega t_0}$ is the transform of $h(t) = \delta(t)+\delta(t-t_0)$ from the prework and $U(j\omega)$ is the transform of a single transmitted pulse $Ap_\tau(t)$. Using linearity and the time shift properties we have

$$Y(j\omega) = A\tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)\left(1 + e^{-j\omega t_0}\right)$$

We are interested in the zeros of $Y(j\omega)$. Solving

$$1 + e^{-j\omega t_0} = 0 \iff \omega t_0 = (2k+1)\pi, \quad k \in \mathbb{Z}$$

So the echo carries no information about the target separation at $\omega = (2k+1)\pi/t_0$, the odd multiples of π/t_0. These are the zeros of $|H(j\omega)| = 2\,|\cos(\omega t_0/2)|$ found in the prework.