Week 5 — Lecture notes

Aperiodic signals and the Fourier transform

1. Quiz 1 review/questions
2. Periodic $\longrightarrow$ aperiodic: motivation
3. Fourier transform as the limit of the Fourier series
4. Fourier transform vs Fourier series: what changed?
5. Fourier transform pairs
6. Properties of the Fourier transform.

Last week: we represented a periodic signal as the sum of sinusoids, using the Fourier series.

This gave us a spectrum: the frequency content of the signal.

But: most signals are not periodic: a radar pulse, a plucked note, a spoken word ...

Question of the lecture: what does the spectrum of a one-off signal look like?

Core idea: we can take the Fourier series of a T-periodic signal and let $T \to \infty$ to get an aperiodic signal.

Periodic $\longrightarrow$ aperiodic: motivation

Let's illustrate this by example: a pulse train, like a radar signal.

Take

$$u_T(t) = A \sum_{m\in\mathbb{Z}} p_\tau(t - mT), \qquad p_\tau(t) = \begin{cases} 1, & |t| < \dfrac{\tau}{2} \\ 0, & |t| > \dfrac{\tau}{2} \end{cases}, \qquad 0 < \tau < T.$$

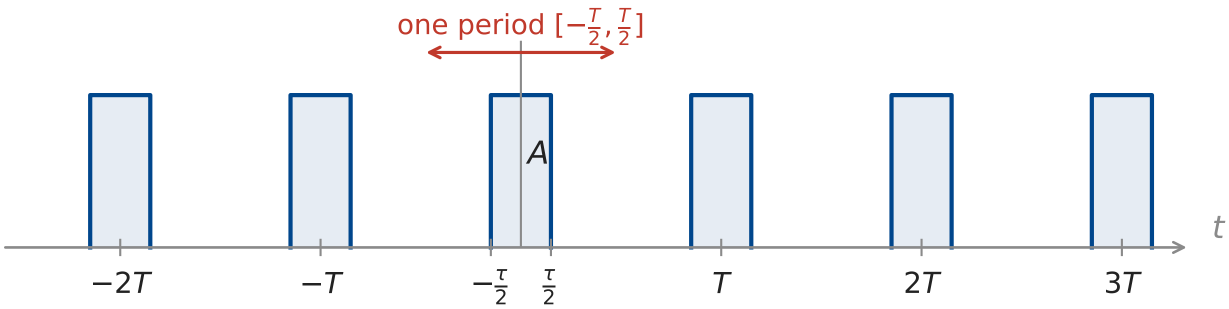

Figure 1: The pulse train $u_T(t)$: amplitude A, pulse width τ, period T. The red bracket marks the period $[-\frac{T}{2}, \frac{T}{2}]$ used for the coefficient integral.

τ varies on time. T varies the period. The ratio $\dfrac{\tau}{T}$ is called the "duty cycle".

Let's calculate the Fourier series coefficients over the red period, $[-\frac{T}{2}, \frac{T}{2}]$. This contains a single pulse.

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_{-T/2}^{T/2} u_T(t)\, e^{-jk\omega_0 t}\,\mathrm{d}t \qquad \left(\omega_0 = \frac{2\pi}{T}\right) \\
&= \frac{A}{T}\int_{-\tau/2}^{\tau/2} e^{-jk\omega_0 t}\,\mathrm{d}t \qquad\qquad \left(T = \frac{2\pi}{\omega_0}\right)
\end{aligned}
$$

$k = 0$:

$$
c_0 = \frac{A}{T}\int_{-\tau/2}^{\tau/2} 1\,\mathrm{d}t = \frac{A\tau}{T}.
$$

$k \neq 0$:

$$
\begin{aligned}
c_k &= \frac{A}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-\tau/2}^{\tau/2} = \frac{A\omega_0}{2\pi}\,\frac{1}{-jk\omega_0}\,\left(e^{-jk\omega_0\frac{\tau}{2}} - e^{jk\omega_0\frac{\tau}{2}}\right) \\
&= \frac{Aj}{2\pi k}\left(-2j\sin\!\left(\frac{k\omega_0\tau}{2}\right)\right) \\
&= \frac{A}{\pi k}\sin\!\left(k\,\frac{\pi\tau}{T}\right) \\
&= A\,\frac{\sin(\pi k d)}{\pi k} \qquad\qquad \left(d = \frac{\tau}{T}\right)
\end{aligned}
$$

$$
\boxed{c_k = A\,d\,\operatorname{sinc}(kd).}
$$

Recall: $\operatorname{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$.

Fourier transform as the limit of the Fourier series

Now, how does this depend on the period T?

$$
\begin{aligned}
c_k &= A\,\frac{\tau}{T}\operatorname{sinc}\!\left(k\,\frac{\tau}{T}\right) \\
&= A\,\frac{\tau}{T}\operatorname{sinc}\!\left(k\,\frac{\tau\omega_0}{2\pi}\right).
\end{aligned}
$$

Call $k\omega_0 = \omega$. Then

$$
Tc_k = A\tau\operatorname{sinc}\!\left(\frac{\tau\omega}{2\pi}\right) =: E(\omega) \qquad \text{(”envelope”)}
$$

Where are the zeros?

$$
\begin{aligned}
\operatorname{sinc}\!\left(\frac{\tau\omega}{2\pi}\right) &= 0 \\
\sin\!\left(\frac{\tau\omega}{2}\right) &= 0 \\
\frac{\tau\omega}{2} &= n\pi \quad \text{for } n \in \mathbb{Z} \\
\omega &= \frac{n2\pi}{\tau}.
\end{aligned}
$$

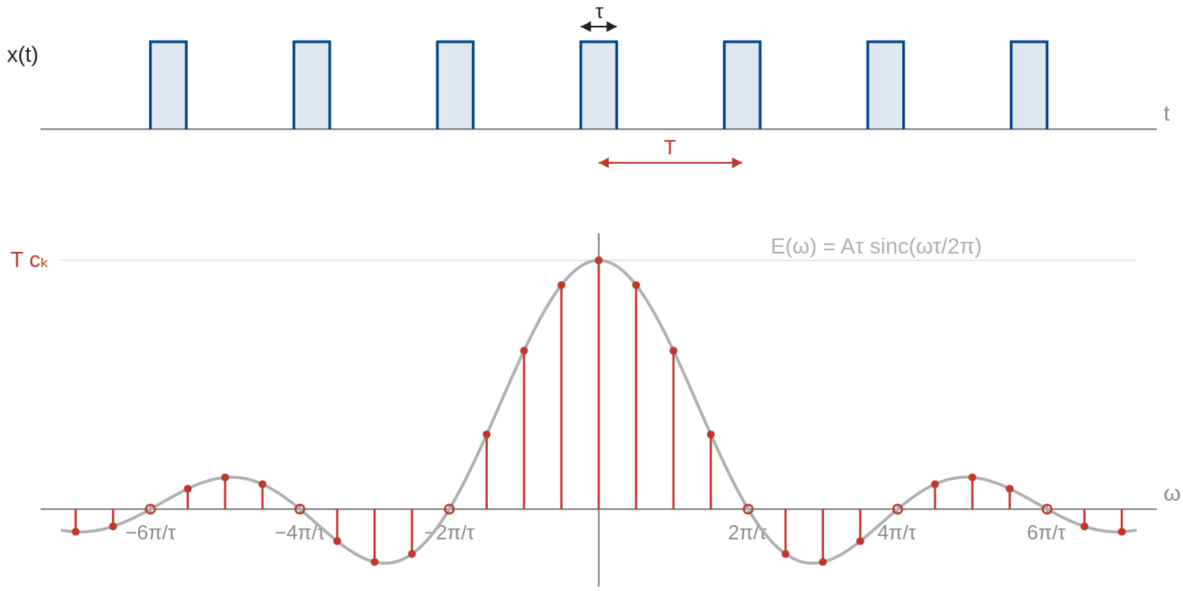

Figure 2: The pulse train (top) and its coefficients (bottom): the lines Tc_k (brick) sampling the fixed envelope $E(\omega) = A\tau\,\mathrm{sinc}(\omega\tau/2\pi)$ (grey) at spacing $\omega_0 = 2\pi/T$. Shown at $T = 4\tau$, in the Tc_k view.

What happens as $T \to \infty$?

$E(\omega)$ doesn't change

Spacing becomes smaller.

What happens to the coefficients c_k?

Let's define $u(t) = \lim_{T\to\infty} u_T(t)$.

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} u_T(t)\, e^{-jk\omega_0 t}\, \mathrm{d}t$$

We define the Fourier transform of $u(t)$ by

$$\boxed{U(j\omega) := \lim_{T\to\infty} Tc_k\Big|_{k\omega_0\to\omega} = \int_{-\infty}^{\infty} u(t)\, e^{-j\omega t}\, \mathrm{d}t.}$$

This can be calculated for any signal for which the integral converges.

We know that the Fourier series can be used to reconstruct the original signal:

$$\boxed{u_T = \sum_{k=-\infty}^{\infty} c_k\phi_k = \sum_{k=-\infty}^{\infty} \langle u_T, \phi_k\rangle_P\, \phi_k, \qquad \phi_k(t) = e^{jk\omega_0 t}}$$

From the week 4 lecture.

Do we get something similar in the Fourier transform limit?

We can rewrite the Fourier series as

$$
\begin{aligned}
u_T(t) &= \sum_{k=-\infty}^{\infty} T c_k \, e^{jk\omega_0 t} \, \frac{1}{T} && \text{[trying to get a “d}\omega\text{” on the RHS]} \\
&= \sum_{k=-\infty}^{\infty} T c_k \, e^{jk\omega_0 t} \, \frac{\omega_0}{2\pi} && [\omega_0 = \text{spacing between samples}] \\
&= \sum_{k=-\infty}^{\infty} T c_k \, e^{j\omega t} \, \frac{\Delta\omega}{2\pi}.
\end{aligned}
$$

Then in the limit,

$$
u(t) = \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} T c_k \, e^{j\omega t} \, \frac{\Delta\omega}{2\pi}.
$$

$$
\boxed{u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(j\omega) \, e^{j\omega t} \, \mathrm{d}\omega} \qquad \text{(the Inverse Fourier transform).}
$$

We have illustrated this above for a pulse chain, but it holds for any signal for which the Fourier transform integral converges $\left(\int_{-\infty}^{\infty} |u(t)| \, \mathrm{d}t < \infty \text{ is sufficient}\right)$.

What has changed versus the Fourier series?

1. Sum $\longmapsto$ integral, discrete index $k \longmapsto$ continuous variable, $k\omega_0 \longmapsto \omega$.
2. Units change. c_k has the same units as u. $U(j\omega)$ has units of $u \times$ time: it is a density. Must be integrated to get anything.

Example: the Kurnell weather radar from the week 2 tutorial.

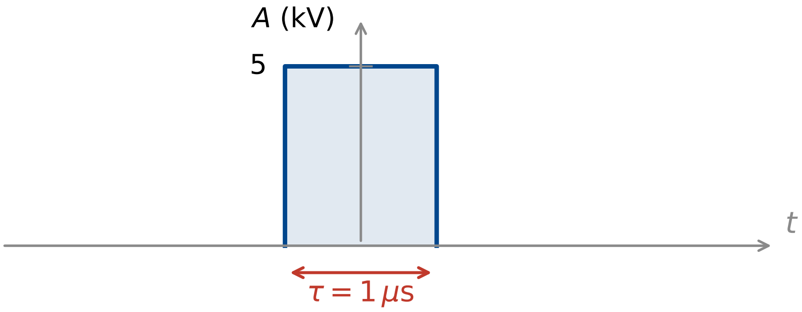

Figure 3: A single transmitted pulse: amplitude 5 kV, width $\tau = 1\,\mu$s.

The parameters τ and T are varied independently:

τ adjusts range resolution:

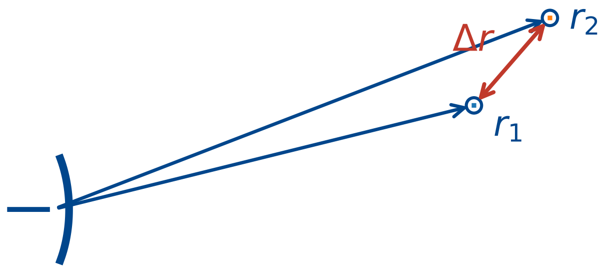

Figure 4: Two targets at ranges r_1 and r_2, separated by Δr.

Consider two targets at ranges r_1 and r_2, with $\Delta r := r_2 - r_1$.

The round trip time for each pulse $t_d = \dfrac{2r}{c}$ $\;(c = 3 \times 10^8\ \mathrm{m\,s^{-1}})$.

$$\Delta t_d = \frac{2\Delta r}{c}.$$

If $\Delta t_d < \tau$, the returning pulses overlap and can't be distinguished.

The minimum distance that can be measured is therefore $\Delta r = \dfrac{c\tau}{2}$. For Kurnell, $\tau = 1\,\mu\mathrm{s}$, so $\Delta r \approx 150\,\mathrm{m}$.

Why not make τ even smaller?

Let's check the spectrum again.

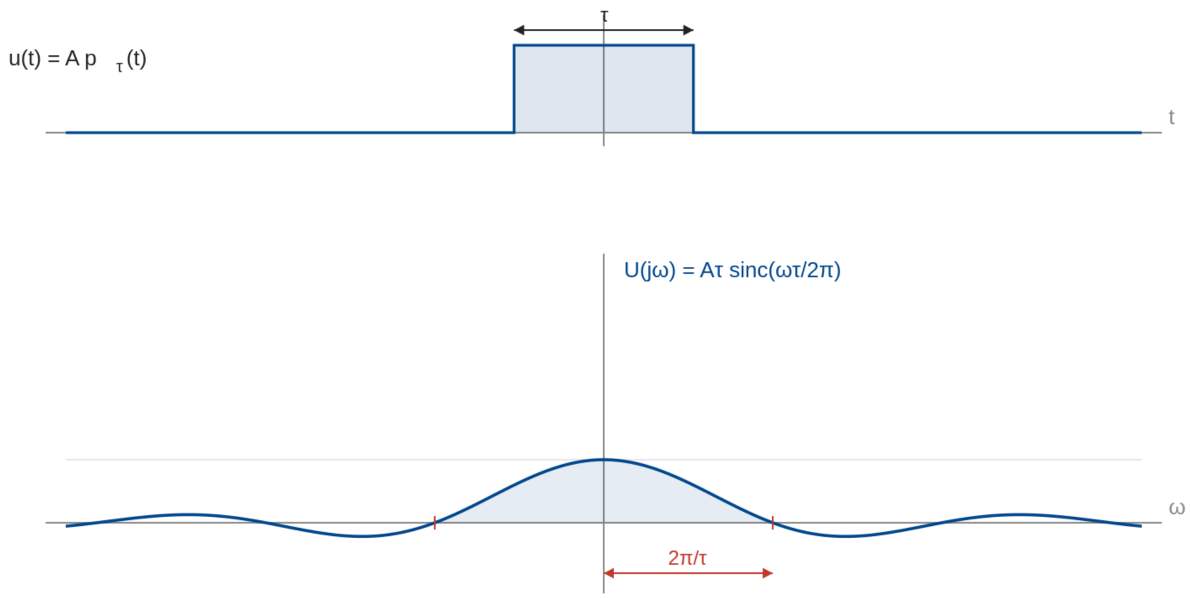

Figure 5: The width-τ pulse $u(t) = Ap_\tau(t)$ (top) and its transform $U(j\omega) = A\tau\,\mathrm{sinc}(\omega\tau/2\pi)$ (bottom), with the main lobe shaded and the first null at $2\pi/\tau$ marked. Shown at $\tau = 1$.

(Higher lobes are filtered — week 10.)

Small τ makes $\dfrac{2\pi}{\tau}$ big, so the lobe spreads and we need more bandwidth to transmit the pulse. Bandwidth is tightly regulated and very expensive!

Fourier transform pairs

The square $\longleftrightarrow$ sinc Fourier transform pair is useful for radar design. Let's derive some other useful pairs.

(a) $u(t) = \delta(t)$.

$$\begin{aligned} U(j\omega) &= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, \mathrm{d}t \\ &= 1 \qquad \text{(sifting property)}. \end{aligned}$$

(b) $u(t) = e^{-at} H(t)$

$$\begin{aligned} U(j\omega) &= \int_{-\infty}^{\infty} e^{-at} H(t)\, e^{-j\omega t}\, \mathrm{d}t \\ &= \int_{0}^{\infty} e^{-(a+j\omega)t}\, \mathrm{d}t \\ &= \left[\frac{-1}{a+j\omega}\, e^{-(a+j\omega)t} \right]_0^{\infty} \\ &= \frac{1}{a+j\omega}. \end{aligned}$$

Extension material

$u(t) = 1$

(c) $u(t) = 1$.

Treat this as the limit of a wide rectangle:

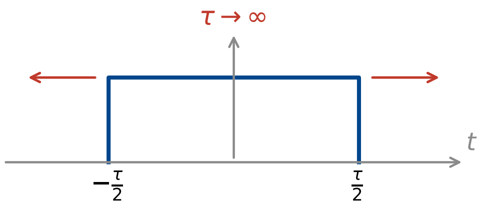

Figure 6: The rectangle widened to a constant.

We already have the Fourier transform of the rectangle: $\tau\,\mathrm{sinc}\!\left(\dfrac{\tau\omega}{2\pi}\right)$.

What is the area under this curve?

$$\begin{aligned} A &= \int_{-\infty}^{\infty} \tau\, \frac{\sin(\frac{\tau\omega}{2})}{\frac{\tau\omega}{2}}\, \mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} \frac{2\sin(\frac{\tau\omega}{2})}{\omega}\, \mathrm{d}\omega. \end{aligned}$$

Let $\mu = \dfrac{\tau\omega}{2} \iff \omega = \dfrac{2\mu}{\tau}$.

$\mathrm{d}\mu = \frac{\tau}{2}\,\mathrm{d}\omega \iff \mathrm{d}\omega = \frac{2\,\mathrm{d}\mu}{\tau}.$

$$
\begin{aligned}
A &= \int_{-\infty}^{\infty} \frac{2\sin(\mu)}{\frac{2\mu}{\tau}}\,\frac{2\,\mathrm{d}\mu}{\tau} \\
&= \int_{-\infty}^{\infty} \frac{2\sin(\mu)}{\mu}\,\mathrm{d}\mu \\
&= \int_{0}^{\infty} \frac{4\sin(\mu)}{\mu}\,\mathrm{d}\mu \qquad \text{(even function).}
\end{aligned}
$$

Fact: $\int_0^{\infty} \frac{\sin(\mu)}{\mu}\,\mathrm{d}\mu = \frac{\pi}{2}$ (full derivation in the notes).

So the Fourier transform of the width τ pulse has constant area 2π.

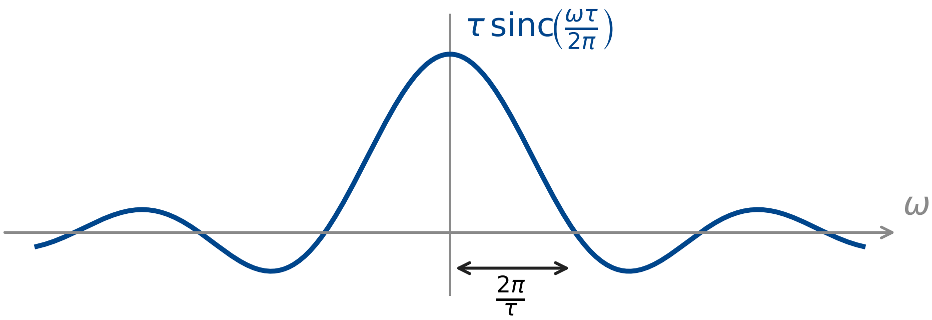

Figure 7: The transform of the width-τ pulse, with the main lobe half-width $\frac{2\pi}{\tau}$ marked.

As we increase τ, the sinc gets taller and narrower.

In the limit $\tau \to \infty$, we have an impulse with area 2π.

So $1 \xrightarrow{\text{FT}} 2\pi\delta(\omega)$.

(d) $e^{j\omega_0 t} \longrightarrow 2\pi\delta(\omega - \omega_0)$

We already have the Fourier transform of 1:

$$\int_{-\infty}^{\infty} 1 \cdot e^{-j\omega t}\,\mathrm{d}t = 2\pi\delta(\omega).$$

$$\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t}\,\mathrm{d}t = \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\,\mathrm{d}t = 2\pi\delta(\omega - \omega_0).$$

So

$$\cos(\omega_0 t) = \tfrac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \overset{\text{FT}}{\rightsquigarrow} \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

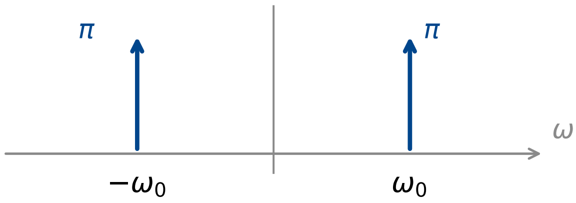

Figure 8: The spectrum of $\cos(\omega_0 t)$: two impulses of weight π at $\pm\omega_0$.

The Fourier series!

Properties of the Fourier transform

This is also a useful general property of the Fourier transform:

$e^{j\omega_0 t}u(t) \;\longleftrightarrow\; U(j(\omega-\omega_0)) \qquad \text{(frequency shift)}$

There are several other useful properties:

$u(t-t_0)$:

$$\int_{-\infty}^{\infty} u(t-t_0)\, e^{-j\omega t}\,\mathrm{d}t = \int_{-\infty}^{\infty} u(\tau)\, e^{-j\omega(\tau+t_0)}\,\mathrm{d}\tau \quad [\tau = t-t_0,\ t=\tau+t_0,\ \mathrm{d}\tau = \mathrm{d}t]$$
$$= e^{-j\omega t_0}\, U(j\omega).$$

$$u(t-t_0) \;\longleftrightarrow\; e^{-j\omega t_0}\, U(j\omega) \qquad \text{(time shift)}.$$

$u(at)$:

$$\int_{-\infty}^{\infty} u(at)\, e^{-j\omega t}\,\mathrm{d}t. \qquad \text{Let } \tau = at,\ \ \mathrm{d}\tau = a\,\mathrm{d}t,\ \ t = \frac{\tau}{a}.$$
$$= \frac{a}{|a|}\int_{-\infty}^{\infty} u(\tau)\, e^{-j\omega\frac{\tau}{a}}\,\frac{\mathrm{d}\tau}{a} \quad \text{[to get the limits the right way]}$$
$$= \frac{1}{|a|}\int_{-\infty}^{\infty} u(\tau)\, e^{-j\frac{\omega}{a}\tau}\,\mathrm{d}\tau$$
$$= \frac{1}{|a|}\,U\!\left(\frac{j\omega}{a}\right).$$

$$u(at) \;\longleftrightarrow\; \frac{1}{|a|}U\!\left(\frac{j\omega}{a}\right) \qquad \text{(time scaling)}.$$

$\alpha u_1(t) + \beta u_2(t) \;\longleftrightarrow\; \alpha U_1(j\omega) + \beta U_2(j\omega) \qquad \text{(linearity)}.$

$y = h * u.$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, u(\tau)\,\mathrm{d}\tau$$

$$Y(j\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t-\tau)u(\tau)\,\mathrm{d}\tau\ e^{-j\omega t}\,\mathrm{d}t$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\,\mathrm{d}t\ u(\tau)\,\mathrm{d}\tau \quad \text{[using Fubini's theorem]}$$

Let $\mu = t - \tau$. $\mathrm{d}t = \mathrm{d}\mu$. $t = \mu + \tau$.

$$\begin{aligned} Y(j\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\mu)\, e^{-j\omega(\mu+\tau)}\,\mathrm{d}\mu\; u(\tau)\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} h(\mu)\, e^{-j\omega\mu}\,\mathrm{d}\mu \int_{-\infty}^{\infty} u(\tau)\, e^{-j\omega\tau}\,\mathrm{d}\tau \end{aligned}$$

$$Y(j\omega) = H(j\omega)\, U(j\omega)$$

$$\boxed{y = h * u \;\longleftrightarrow\; Y = HU} \qquad \text{(convolution)}$$

This is the foundation of the rest of the course.

$\dfrac{\mathrm{d}}{\mathrm{d}t}u(t) \;\rightsquigarrow\; \displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}u(t)\, e^{-j\omega t}\,\mathrm{d}t$

Integrate by parts: $\displaystyle\int u'v = uv - \int uv'$

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}u(t)\, e^{-j\omega t}\,\mathrm{d}t &= \underbrace{\left[u(t)e^{-j\omega t}\right]_{-\infty}^{\infty}}_{=0 \text{ if } u(-\infty)=u(\infty)=0} + j\omega \int_{-\infty}^{\infty} u(t)\, e^{-j\omega t}\,\mathrm{d}t \\ &= j\omega\, U(j\omega). \end{aligned}$$

So

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}u(t) \;\longleftrightarrow\; j\omega\, U(j\omega).}$$

Summary

Transform pairs.

$u(t)$	$U(j\omega)$	derived in
$A\,p_\tau(t)$	$A\tau\,\mathrm{sinc}\left(\dfrac{\omega\tau}{2\pi}\right)$	§2
$\delta(t)$	1	(a)
$e^{-at}H(t)$	$\dfrac{1}{a+j\omega}$	(b)
1	$2\pi\delta(\omega)$	(c)
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	(d)
$\cos(\omega_0 t)$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$	(d)

Properties.

	$u(t)$	$U(j\omega)$
frequency shift	$e^{j\omega_0 t}u(t)$	$U\big(j(\omega-\omega_0)\big)$
time shift	$u(t-t_0)$	$e^{-j\omega t_0}U(j\omega)$
time scaling	$u(at)$	$\dfrac{1}{\lvert a\rvert}U\left(\dfrac{j\omega}{a}\right)$
linearity	$\alpha u_1(t)+\beta u_2(t)$	$\alpha U_1(j\omega)+\beta U_2(j\omega)$
convolution	$y = h*u$	$Y = HU$
differentiation	$\dfrac{\mathrm{d}}{\mathrm{d}t}u(t)$	$j\omega\, U(j\omega)$
duality	$X(jt)$	$2\pi x(-\omega)$

	$u(t)$	$U(j\omega)$
Parseval	$\int_{-\infty}^{\infty} u(t)\overline{y(t)}\,\mathrm{d}t$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} U(j\omega)\overline{Y(j\omega)}\,\mathrm{d}\omega$

Other properties

Duality: if $x(t) \rightsquigarrow X(j\omega)$, then

$$X(jt) \;\rightsquigarrow\; 2\pi x(-\omega).$$

Proof:

$$\begin{aligned} x(-\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(js)\, e^{-j\omega s}\,\mathrm{d}s \\ &= \frac{1}{2\pi}\,\mathcal{F}\{X(jt)\}(\omega). \end{aligned}$$

Example: we know $\delta(t) \rightsquigarrow 1$.

The dual gives us another pair: $1 \rightsquigarrow 2\pi\delta(\omega)$.

Parseval's theorem:

$$\int_{-\infty}^{\infty} u(t)\,\overline{y(t)}\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(j\omega)\,\overline{Y(j\omega)}\,\mathrm{d}\omega.$$

Special case: if $y = u$:

$$\int_{-\infty}^{\infty} |u(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |U(j\omega)|^2\,\mathrm{d}\omega.$$

Proof:

$$\begin{aligned} \int u\bar{y}\,\mathrm{d}t &= \int u(t)\,\frac{1}{2\pi}\int \overline{Y(j\omega)\,e^{j\omega t}}\,\mathrm{d}\omega\;\mathrm{d}t \\ &= \frac{1}{2\pi}\int \left[\int u(t)\,e^{-j\omega t}\,\mathrm{d}t\right]\overline{Y(j\omega)}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int U(j\omega)\,\overline{Y(j\omega)}\,\mathrm{d}\omega. \end{aligned}$$

Example: the RLC circuit

Example: RLC circuit with v_C as output.

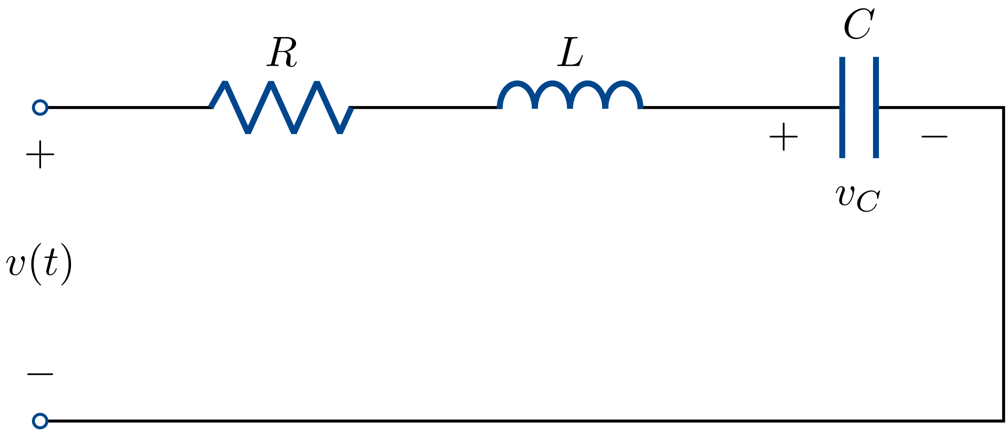

Figure 9: Series RLC circuit with v_C as the output (drawn with circuitikz).

We know the impulse response is

$$g(t) = 6.25\, e^{-3t} \sin(4t)\, H(t).$$

What is the spectrum of this signal?

$$\sin(4t) = \frac{1}{2j}\left(e^{j4t} - e^{-j4t}\right)$$

So

$$g(t) = \frac{6.25}{2j}\left(e^{(-3+4j)t} - e^{(-3-4j)t}\right).$$

Use the Fourier transform pair $e^{-at}H(t) = \dfrac{1}{a + j\omega}$:

$$\begin{aligned}
G(j\omega) &= -3.125j\left(\frac{1}{-(-3+4j)+j\omega} - \frac{1}{-(-3-4j)+j\omega}\right) \\
&= -3.125j\left(\frac{(+3+4j+j\omega)-(+3-4j+j\omega)}{(+3-4j+j\omega)(+3+4j+j\omega)}\right) \\
&= -3.125j\left(\frac{8j}{(+3-4j)(+3+4j)+j\omega(+3-4j)+(+3+4j)j\omega-\omega^2}\right) \\
&= \frac{25}{-\omega^2+6j\omega+25}.
\end{aligned}$$

What does the spectrum look like?

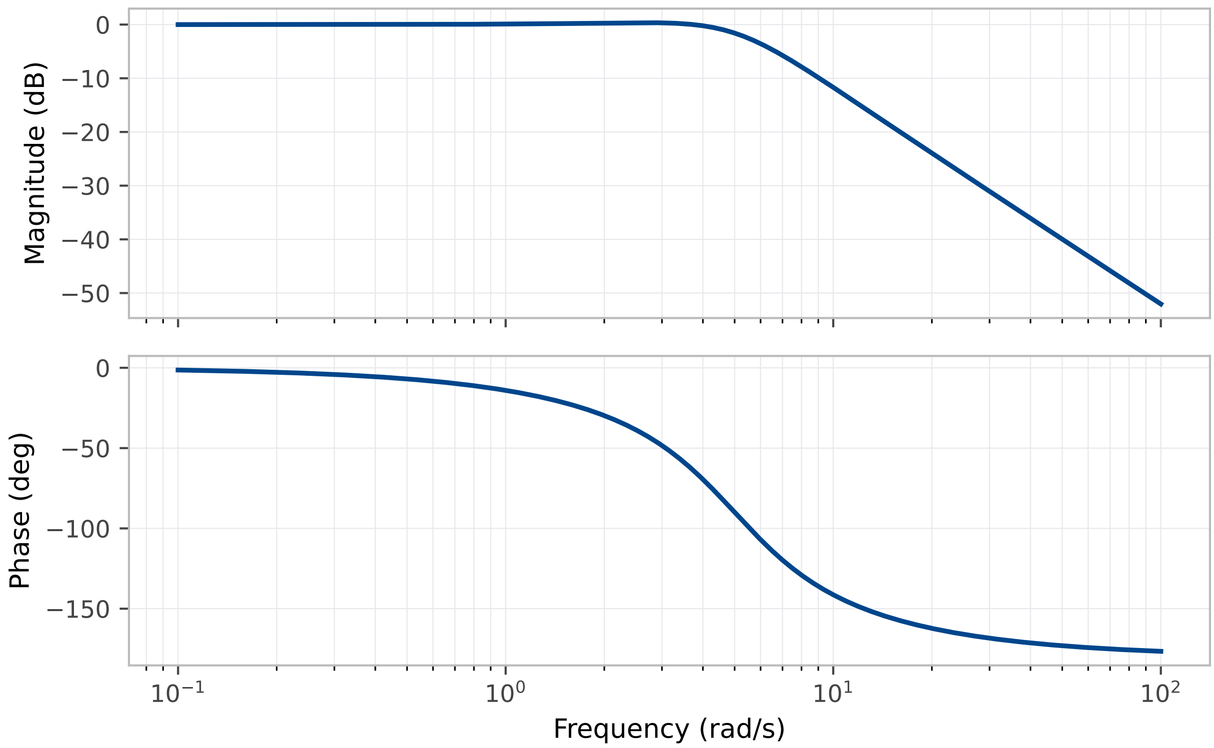

Figure 10: Magnitude and phase of $G(j\omega) = \dfrac{25}{-\omega^2 + 6j\omega + 25}$.