Signals and Systems — Tutorial 3

Fourier Series

Tutorial information

- It is expected that you attempt the prework questions before coming to the tutorial each week and submit your *handwritten* solutions to your tutor within the first 15 minutes of your tutorial.
- Not all questions are equally difficult. Difficult questions are marked with a $\star$.
- Questions marked **Extension** go beyond the assessable content for this course and are provided to give a deeper theoretical background and understanding.
- You may not be able to complete all questions within the tutorial time. It is *strongly* recommended that you attempt all questions in your own time and ask your tutor if you are stuck.

Prework questions

A: Frequency and Spectra

Suppose $f(t) = \sin(t) + \cos(3t)$ find the fundamental frequency of f and plot its magnitude and phase spectrum.

B: Fourier Series

Determine the complex Fourier series for 1. $f(t) = \sin(\omega t) + \cos(\omega t)$ and 2. $f(t) = \sin(\omega t)\cos(\omega t)$

C: MATLAB

Suppose $f(t)$ has a Fourier expansion

$$f(t) = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \sin\left((k+1)t\right)$$

1. Without calculation or plotting is $f(t)$ even, odd or neither? Explain.
2. Using MATLAB (or your favourite programming language) plot $f(t)$ for $k = 0, 1, \ldots, 5$. What function is $f(t)$? *You do not have to include a copy of the plot in your prework submission.*

Tutorial questions

1: Fourier Series

Find the Fourier series of **(a)** each of the periodic signals in Figure 1 (**Hint:** Use properties of the Fourier series) and **(b)** of the rectified sine-wave $f(t) = |\sin(100t)|$ which was the output from the *'full-wave rectifier'* in Question 5 in Tutorial 1.

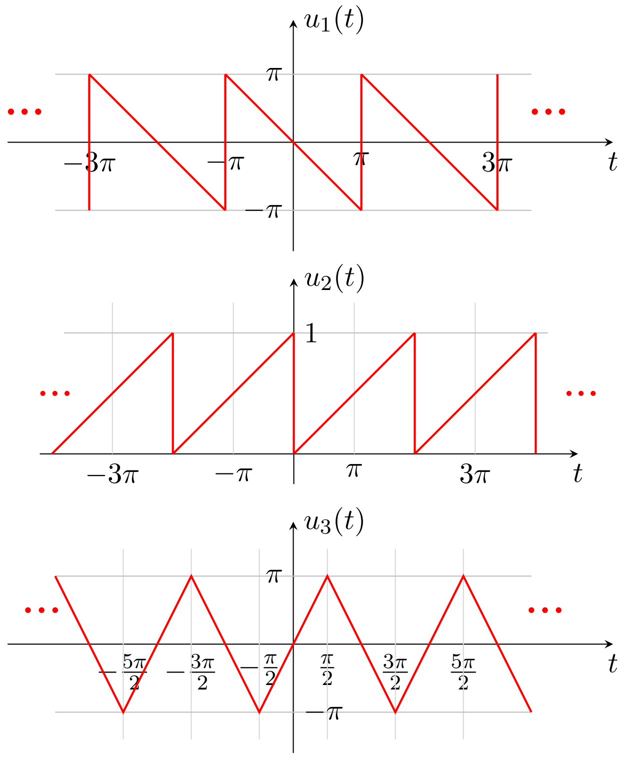

Figure 1: Periodic Signals

2: Exponential and Trigonometric Series

1. From the definitions given in the lecture notes, find both the exponential and trigonometric Fourier series for at least one of the periodic signals given in Figure 1.

2. Use Euler's formula to prove that

   (a) $c_0 = a_0$,

   (b) $|c_k| = \frac{1}{2}\sqrt{a_k^2 + b_k^2}$,

   (c) $2c_k = a_k - jb_k$ and

   (d) $\bar{c}_k = c_{-k}$, where $\bar{c}_k$ is the complex conjugate of the coefficient c_k.

   and verify the above identities for the coefficients a_k, b_k and c_k of the series found in Part 1.

3. Recall from Lecture 4 that the Fourier series for the square-wave shown in Figure 2 is

$$u(t) = \sum_{k \text{ odd}} 2\,\mathrm{sinc}(k/2)\cos(kt)$$

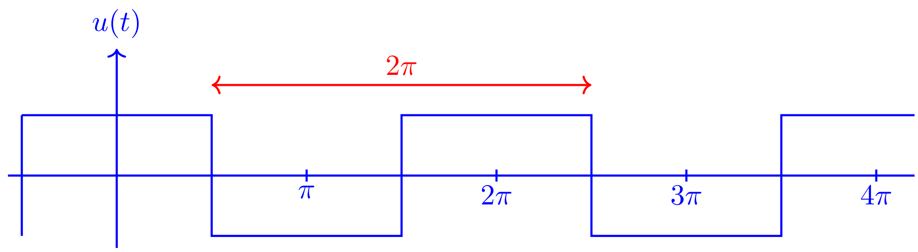

Figure 2: Square-wave with period $T = 2\pi$ and amplitude 1

Consider the periodic square wave $f(t)$ with period $T = 2\pi$ and amplitude ± 1. Compute the Fourier series coefficients c_k for $k = 0, 1, 2, 3$

3: Spectra

1. Plot the magnitude and phase spectrum for at least one of the signals in Figure 1.
2. For the spectrum in Figure 3 find the Fourier expansion and determine if the function is odd, even or neither **Hint:** Consider the spectra of $\sin(\omega t)$ and $\cos(\omega t)$.

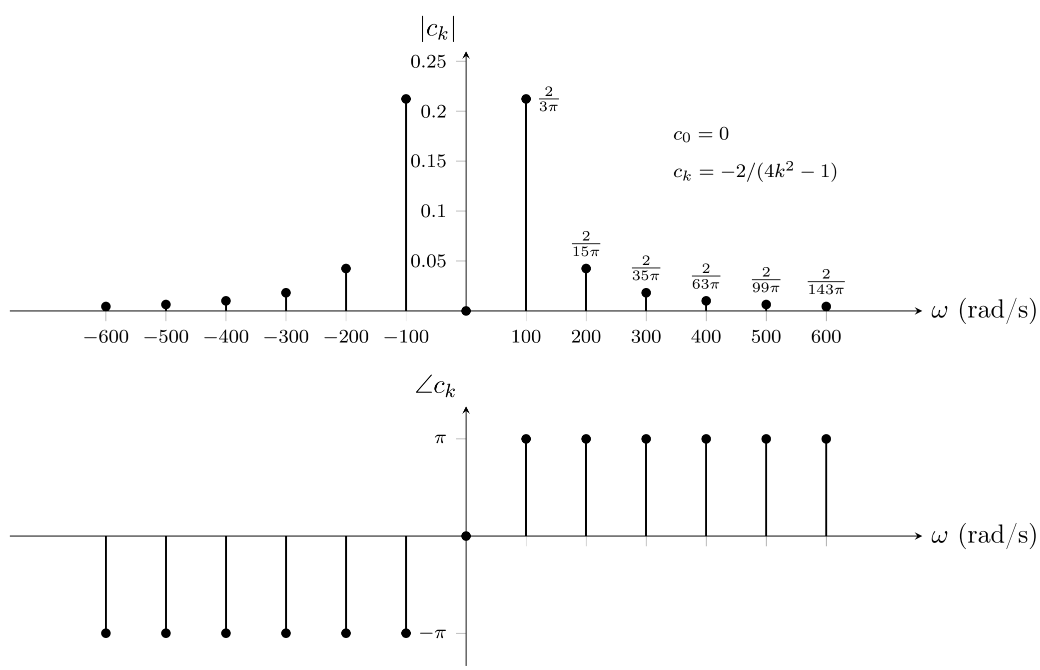

Figure 3: Magnitude and phase spectrum for an unknown signal.

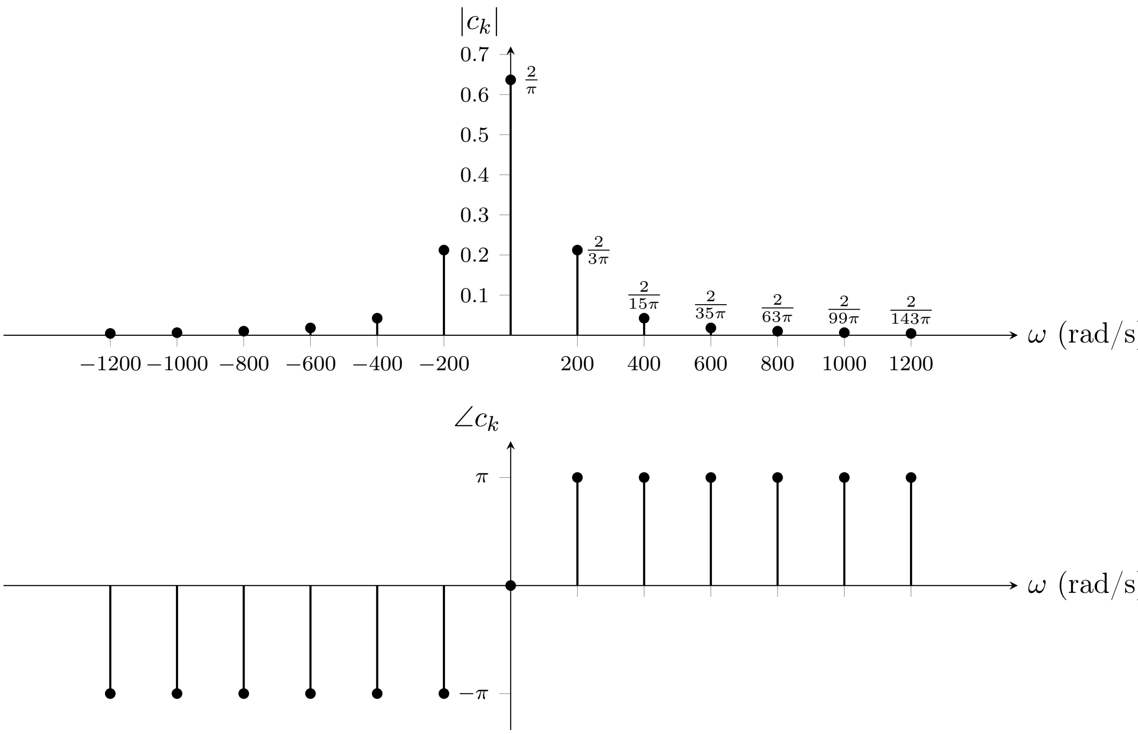

Figure 4: Magnitude and phase spectrum for $|sin(100t)|$.

3. Given the magnitude and phase spectrum of $f(t) = |\sin(100t)|$ is as shown in Figure 4 determine which function has the spectrum shown in Figure 3.

4: Parseval's Theorem

1. Given that the area of a polar function $r(\theta)$ is

$$A = \frac{1}{2}\int_a^b r^2(\theta)\, d\theta$$

Find the area of (a) $f(t) = \cos(t) + \sin(t)$ and (b) $f(t) = 1 - \cos(t)$

$\star$ 2. **[Extension]** Using the Fourier expansion of $f(t) = t$ (see the prework questions). Prove that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

5: Differentiation and Fourier Series

1. **[Extension]** Consider the circuit in Figure 5

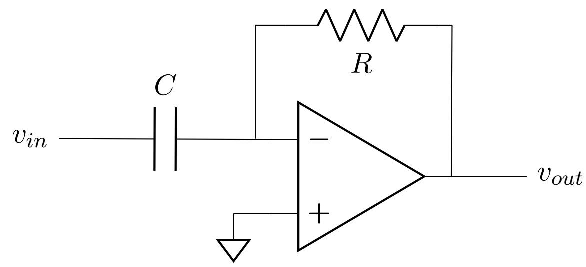

Figure 5: Ideal differentiator

Verify that the input $u(t) = v_{in}(t)$ and output $y(t) = v_{out}(t)$ satisfy

$$y(t) = -RC\frac{d}{dt}u(t)$$

2. Assume $RC = 1$. Suppose $u(t) = u_3(t)$ is the triangular wave given in Figure 1. Verify that $y(t)$ is a square-wave with amplitude 1 and the same frequency as the square-wave given in Figure 2.

3. Consider the Fourier series for a triangular wave $u_3(t)$ and square-wave in Figure 2 i.e.

$$u_3(t) = \sum_k c_k e^{jk\omega t} \text{ and } u_{sq}(t) = \sum_k d_k e^{j\omega kt}$$

   (a) By differentiating term-wise, verify that $u_3'(t) = 2u_{sq}(t)$

   (b) **[Extension]** Find d_k/c_k for each k. How does this compare with the frequency response function (equivalent impedence), $Z_{eq} = -j\omega RC$, of the circuit in Figure 5?

⋆ 4. Find the Fourier series for the signal $u_\delta(t) = \delta(t - n\pi)$ shown in Figure 6 where $n \in \mathbb{Z}$.

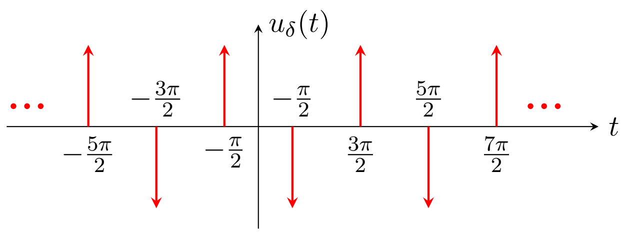

Figure 6: $\delta(t - n\pi)$