Signals and Systems — Tutorial 3

Fourier Series

[Tutor version — includes solutions]

Tutorial information

- It is expected that you attempt the prework questions before coming to the tutorial each week and submit your *handwritten* solutions to your tutor within the first 15 minutes of your tutorial.
- Not all questions are equally difficult. Difficult questions are marked with a $\star$.
- Questions marked **Extension** go beyond the assessable content for this course and are provided to give a deeper theoretical background and understanding.
- You may not be able to complete all questions within the tutorial time. It is *strongly* recommended that you attempt all questions in your own time and ask your tutor if you are stuck.

Prework questions

A: Frequency and Spectra

Suppose $f(t) = \sin(t) + \cos(3t)$ find the fundamental frequency of f and plot its magnitude and phase spectrum.

B: Fourier Series

Determine the complex Fourier series for 1. $f(t) = \sin(\omega t) + \cos(\omega t)$ and 2. $f(t) = \sin(\omega t)\cos(\omega t)$

C: MATLAB

Suppose $f(t)$ has a Fourier expansion

$$f(t) = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \sin\left((k+1)t\right)$$

1. Without calculation or plotting is $f(t)$ even, odd or neither? Explain.
2. Using MATLAB (or your favourite programming language) plot $f(t)$ for $k = 0, 1, \ldots, 5$. What function is $f(t)$? *You do not have to include a copy of the plot in your prework submission.*

Tutorial questions

1: Fourier Series

Find the Fourier series of **(a)** each of the periodic signals in Figure 1 (**Hint:** Use properties of the Fourier series) and **(b)** of the rectified sine-wave $f(t) = |\sin(100t)|$ which was the output from the *'full-wave rectifier'* in Question 5 in Tutorial 1.

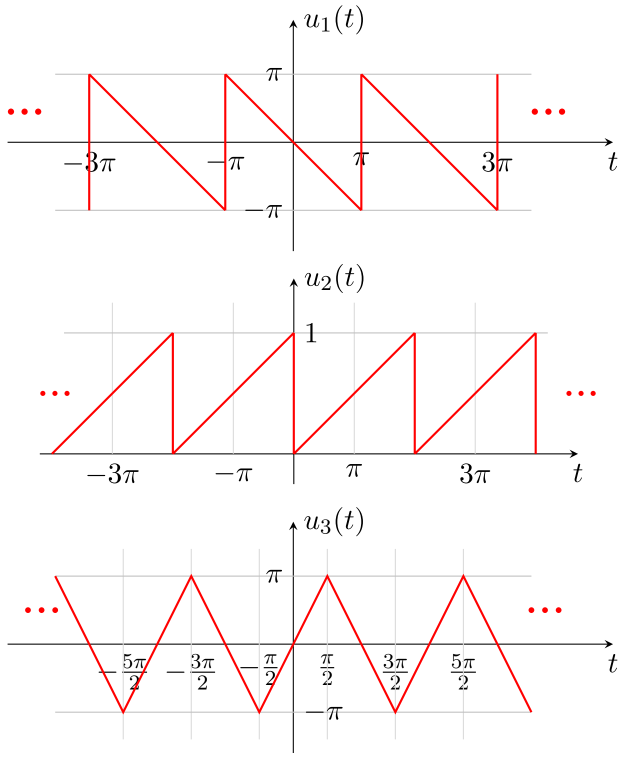


Figure 1: Periodic Signals

1. Signal is periodic with fundamental $T = 2\pi$. Fourier series will be given by

$$u_1(t) = \sum_k c_k e^{jkt} = a_0 + \sum_{k>0} a_k \cos(kt) + b_k \sin(kt)$$

Notice the function is odd so we expect $a_k = 0$ and c_k is imaginary for all k. To find the coefficients

$$\begin{aligned}
c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u_1(t)e^{-jkt}\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} -te^{-jkt}\,dt \\
&= \frac{1}{2\pi j}\int_{-\pi}^{\pi} \frac{d}{dk}e^{-jkt}\,dt \\
&= \frac{1}{2\pi j}\frac{d}{dk}\int_{-\pi}^{\pi} e^{-jkt}\,dt \\
&= \frac{1}{2\pi j}\frac{d}{dk}\left(\frac{e^{-jk\pi}-e^{jk\pi}}{-jk}\right) \\
&= \frac{1}{2\pi j}\frac{d}{dk}\left(\frac{2\sin(k\pi)}{k}\right) \\
&= \frac{1}{\pi j}\left(\frac{\pi k\cos(k\pi)-\sin(k\pi)}{k^2}\right) \\
&= \frac{(-1)^k}{jk}
\end{aligned}$$

with $c_0 = 0$, since $u_1(t)$ has zero mean. So we have the *complex* Fourier series

$$\sum_{k\neq 0}\left(\frac{(-1)^k}{jk}\right)e^{jkt}$$

Let's find a_k and b_k from first principles

$$\begin{aligned}
a_k &= \frac{2}{2\pi}\int_{-\pi}^{\pi} u_1(t)\cos(kt)\,dt \\
&= \frac{1}{\pi}\frac{d}{dk}\int_{-\pi}^{\pi} -\sin(kt)\,dt \\
&= \frac{1}{\pi}\left(\frac{d}{dk}\frac{\cos(k\pi)}{k} - \frac{d}{dk}\frac{\cos(k\pi)}{k}\right) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} -t\sin(kt)\,dt \\ &= \frac{1}{\pi}\frac{d}{dk}\int_{-\pi}^{\pi}\cos(kt)\,dt \\ &= \frac{2}{\pi}\left(\frac{d}{dk}\frac{\sin(k\pi)}{k}\right) \\ &= \frac{2}{\pi}\left(\frac{k\pi\cos(k\pi)-\sin(k\pi)}{k^2}\right) \\ &= \frac{2}{k}\cos(k\pi) \\ &= \frac{2(-1)^k}{k} \end{aligned}$$

Giving the trigonometric series

$$u_1(t) = \sum_{k>0}\frac{2}{k}(-1)^k\sin(kt)$$

2. Note that $u_2 = 0.5 - u_1(t-\pi)/(2\pi)$. We use the time-shift, scaling and linearity of the transform to obtain

$$\begin{aligned} u_2(t) &= \frac{1}{2} - \frac{1}{2\pi}\sum_{k\neq 0}\frac{(-1)^k}{jk}e^{j(kt-\pi k)} \\ &= \frac{1}{2} + \sum_{k\neq 0}\frac{j}{2\pi k}e^{jkt} \end{aligned}$$

3. $u_3(t)$ is odd, so $a_k = 0$ and c_k is imaginary. Integrating by parts, and using $u_3(0) = u_3(\pi) = 0$,

$$\begin{aligned} b_k &= \frac{2}{\pi}\int_0^{\pi} u_3(t)\sin(kt)\,dt \\ &= \frac{2}{\pi}\left(\left[\frac{-u_3(t)\cos(kt)}{k}\right]_0^{\pi} + \frac{1}{k}\int_0^{\pi}\dot{u}_3(t)\cos(kt)\,dt\right) \\ &= \frac{2}{\pi k}\left(2\int_0^{\pi/2}\cos(kt)\,dt - 2\int_{\pi/2}^{\pi}\cos(kt)\,dt\right) \\ &= \frac{4}{\pi k^2}\left(\sin\left(\frac{k\pi}{2}\right) - \sin(k\pi) + \sin\left(\frac{k\pi}{2}\right)\right) \\ &= \frac{8}{\pi k^2}\sin\left(\frac{k\pi}{2}\right) \end{aligned}$$

which is zero for even k. So

$$u_3(t) = \frac{8}{\pi} \sum_{k \text{ odd}, k>0} \frac{\sin(k\pi/2)}{k^2} \sin(kt) = \frac{8}{\pi} \left(\sin(t) - \frac{\sin(3t)}{9} + \frac{\sin(5t)}{25} - \cdots \right)$$

and $c_k = -jb_k/2$, i.e. for odd k

$$c_k = \frac{-4j}{\pi k^2} \sin\left(\frac{k\pi}{2}\right) = \frac{4}{\pi k^2} e^{-jk\pi/2}$$

4. $f(t) = |\sin(100t)|$. Simplify by substituting $a = 100$ so $f(t) = |\sin(at)|$. Note that $f(t)$ has fundamental period $T = \pi/a$. To find c_k

$$\begin{aligned}
c_k &= \langle f(t), e^{j2akt} \rangle \\
&= \frac{1}{T} \int_0^T |\sin(at)| e^{-j2akt}\, dt \\
&= \frac{1}{T} \int_0^T \sin(at) e^{-j2akt}\, dt \\
&= \frac{1}{T} \int_0^T \frac{e^{jat} - e^{-jat}}{2j} e^{-j2akt}\, dt \\
&= \frac{1}{2jT} \int_0^T e^{jat(1-2k)} - e^{-jat(1+2k)}\, dt \\
&= \frac{1}{2jT} \left(\frac{e^{jat(1-2k)}}{ja(1-2k)} + \frac{e^{-jat(1+2k)}}{ja(1+2k)} \right) \Bigg|_0^T \\
&= \frac{-1}{2\pi} \left(\frac{e^{jat(1-2k)}}{(1-2k)} + \frac{e^{-jat(1+2k)}}{(1+2k)} \right) \Bigg|_0^T \\
&= \frac{-1}{2\pi} \left(\frac{e^{j\pi(1-2k)} - 1}{(1-2k)} + \frac{e^{-j\pi(1+2k)} - 1}{(1+2k)} \right) \\
&= \frac{-1}{2\pi} \left(\frac{-2}{1-2k} + \frac{-2}{1+2k} \right) \\
&= \frac{-1}{2\pi} \left(\frac{-2(1+2k) - 2(1-2k)}{(1-2k)(1+2k)} \right) \\
&= \frac{-1}{2\pi} \left(\frac{-4}{1-4k^2} \right) \\
&= \frac{-2}{\pi(4k^2 - 1)}
\end{aligned}$$

So the Fourier series is

$$f(t) = \sum_k \frac{-2e^{j2akt}}{\pi(4k^2-1)} = \sum_k \frac{-2e^{j200kt}}{\pi(4k^2-1)}$$

2: Exponential and Trigonometric Series

1. From the definitions given in the lecture notes, find both the exponential and trigonometric Fourier series for at least one of the periodic signals given in Figure 1.

2. Use Euler's formula to prove that

   (a) $c_0 = a_0$,

   (b) $|c_k| = \frac{1}{2}\sqrt{a_k^2 + b_k^2}$,

   (c) $2c_k = a_k - jb_k$ and

   (d) $\bar{c}_k = c_{-k}$, where $\bar{c}_k$ is the complex conjugate of the coefficient c_k.

   and verify the above identities for the coefficients a_k, b_k and c_k of the series found in Part 1.

3. Recall from Lecture 4 that the Fourier series for the square-wave shown in Figure 2 is

$$u_{sq}(t) = \sum_{k \text{ odd}} 2\,\mathrm{sinc}(k/2)\cos(kt)$$

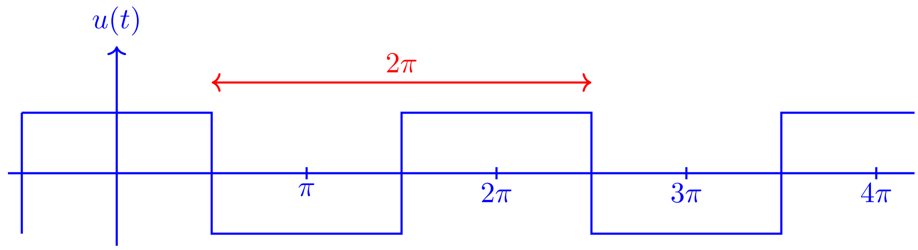


Figure 2: Square-wave with period $T = 2\pi$ and amplitude 1

Consider the periodic square wave $f(t)$ with period $T = 2\pi$ and amplitude ± 1. Compute the Fourier series coefficients c_k for $k = 0, 1, 2, 3$

1. Take $u_2(t) = t/(2\pi)$ on $(0, 2\pi)$, so $T = 2\pi$ and $\omega = 1$. From the definitions

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\,dt = \frac{1}{4\pi^2}\left(2\pi^2\right) = \frac{1}{2}$$

   and for $k \neq 0$, integrating by parts,

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} \frac{t}{2\pi} e^{-jkt}\,dt = \frac{1}{4\pi^2}\left(\left[\frac{te^{-jkt}}{-jk}\right]_0^{2\pi} + \frac{1}{jk}\int_0^{2\pi} e^{-jkt}\,dt\right)$$
$$= \frac{1}{4\pi^2}\left(\frac{2\pi}{-jk}\right) = \frac{j}{2\pi k}$$

which agrees with Part 2 of Question 1. Similarly

$$a_k = \frac{1}{\pi}\int_0^{2\pi} \frac{t}{2\pi}\cos(kt)\,dt = \frac{1}{2\pi^2}\left[\frac{t\sin(kt)}{k} + \frac{\cos(kt)}{k^2}\right]_0^{2\pi} = 0$$
$$b_k = \frac{1}{\pi}\int_0^{2\pi} \frac{t}{2\pi}\sin(kt)\,dt = \frac{1}{2\pi^2}\left[\frac{-t\cos(kt)}{k} + \frac{\sin(kt)}{k^2}\right]_0^{2\pi} = \frac{-1}{\pi k}$$

Giving the two series

$$u_2(t) = \frac{1}{2} + \sum_{k\neq 0}\frac{j}{2\pi k}e^{jkt} = \frac{1}{2} - \sum_{k>0}\frac{1}{\pi k}\sin(kt)$$

Here $a_0 = c_0 = 1/2$ and, for $k > 0$, $2c_k = a_k - jb_k = j/(\pi k)$, as in Part 2.

2. This follows similarly to the derivation given in lectures. This question is mainly to reinforce the connection between the series. To avoid redoing the lecture derivation. Define

$$c_k = \langle f(t), e^{j\omega kt}\rangle = \frac{1}{T}\int_0^T f(t)e^{-j\omega kt}\,dt = \frac{1}{T}\int_0^T f(t)\left(\cos(\omega kt) - j\sin(\omega kt)\right)\,dt$$

and for $k \neq 0$

$$a_k = \frac{2}{T}\int_0^T f(t)\cos(\omega kt)\,dt \text{ and } b_k = \frac{2}{T}\int_0^T f(t)\sin(\omega kt)\,dt$$

Consider for $k \neq 0$

$$c_k + c_{-k} = \frac{2}{T}\int_0^T f(t)\cos(\omega kt)\,dt = a_k \text{ and } c_k - c_{-k} = -\frac{2j}{T}\int_0^T f(t)\sin(\omega kt)\,dt = -jb_k$$

Thus $2c_k = a_k - jb_k$ and $2c_{-k} = a_k + jb_k$, which is **(c)**. As $f(t)$ is real, a_k and b_k are real, so $2|c_k| = \sqrt{a_k^2 + b_k^2}$, which is **(b)**, and $\bar{c}_k = \frac{1}{2}(a_k + jb_k) = c_{-k}$, which is **(d)**. To see $c_0 = a_0$ note that for

$$c_0 + \sum_{k\neq 0} c_k e^{j\omega kt} = a_0 + \sum_{k\geq 1} a_k\cos(\omega kt) + \sum_{k\neq 0} b_k\sin(\omega kt)$$

Are equal if and only if the constant terms are equal.

3. Note that the series for u_{sq} is given as a trigonometric series and $b_k = 0$ for all k. The sum is over odd k only, so $a_k = 0$ for every even k, including $k = 0$: the series has no constant term. Hence $c_0 = a_0 = 0$, $c_k = 0$ for even k, and for odd k, $c_k = a_k/2 = \operatorname{sinc}\left(\frac{k}{2}\right) = \frac{\sin(\pi k/2)}{\pi k/2}$. Computing for $k = 0, 1, 2$ and 3 we have

$$c_0 = 0, c_1 = \frac{2}{\pi}, c_2 = 0 \text{ and } c_3 = -\frac{2}{3\pi}$$

3: Spectra

1. Plot the magnitude and phase spectrum for at least one of the signals in Figure 1.
2. For the spectrum in Figure 3 find the Fourier expansion and determine if the function is odd, even or neither **Hint:** Consider the spectra of $\sin(\omega t)$ and $\cos(\omega t)$.

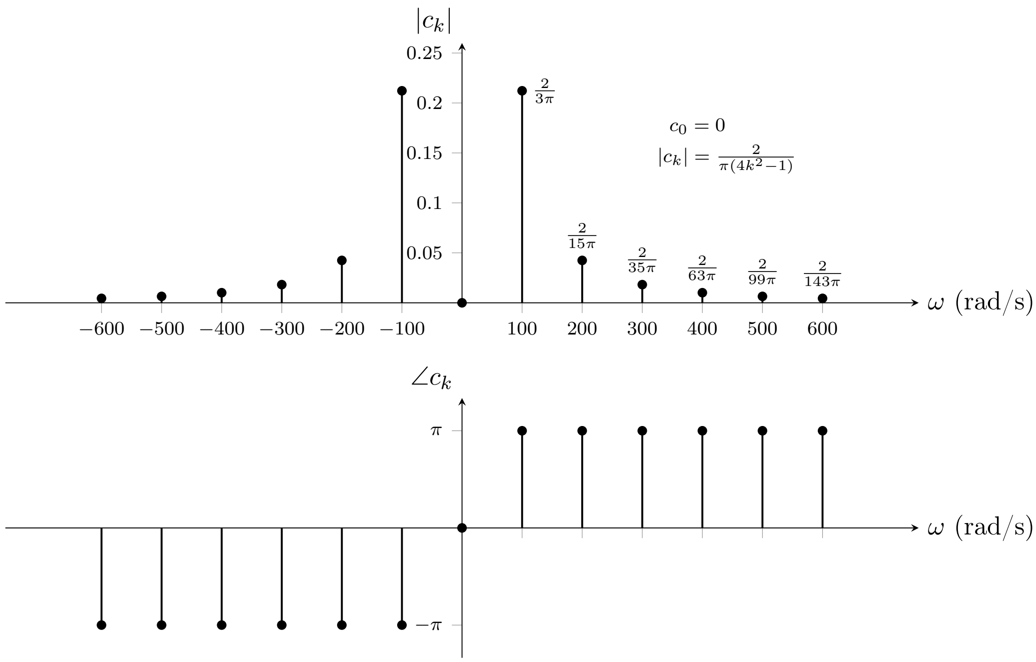


Figure 3: Magnitude and phase spectrum for an unknown signal.

3. Given the magnitude and phase spectrum of $f(t) = |\sin(100t)|$ is as shown in Figure 4 determine which function has the spectrum shown in Figure 3.

1. Take $u_2(t)$, with $c_0 = 1/2$ and $c_k = j/(2\pi k)$ from Question 2. As $\omega_0 = 1$ rad/s the lines lie at $\omega = k$, with

$$|c_0| = \frac{1}{2}, \quad \angle c_0 = 0, \quad |c_k| = \frac{1}{2\pi|k|}, \quad \angle c_k = \begin{cases} \pi/2, & k > 0 \\ -\pi/2, & k < 0 \end{cases}$$

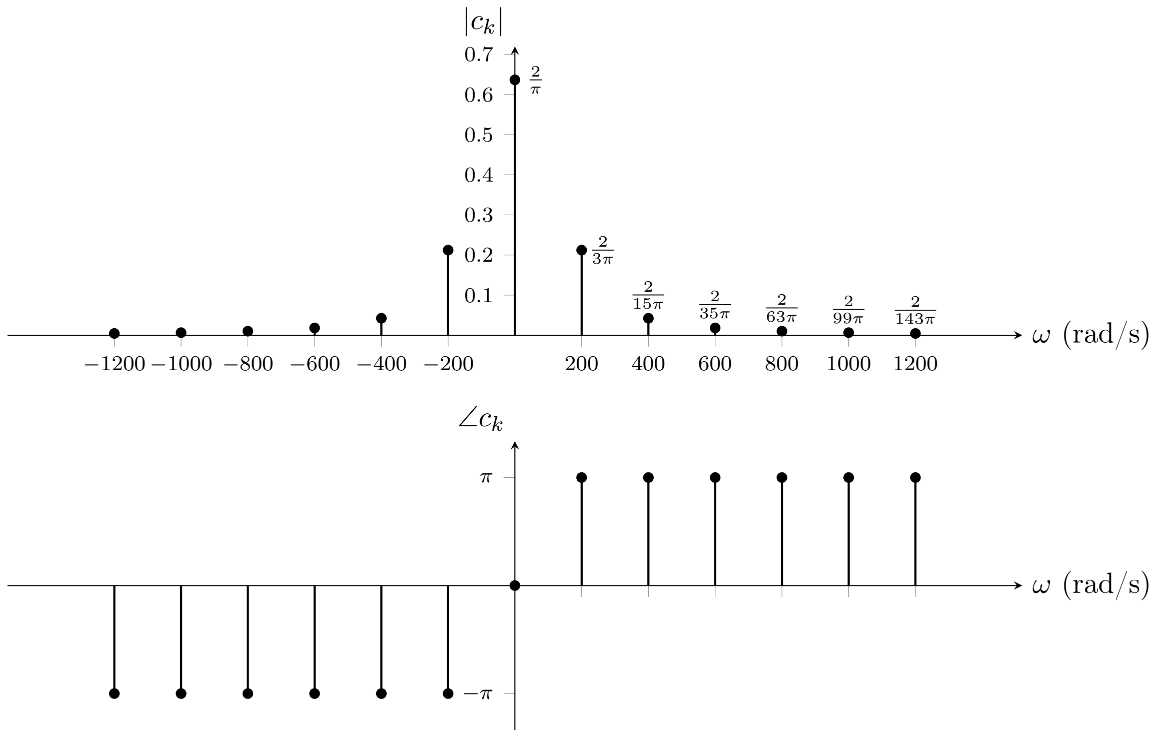
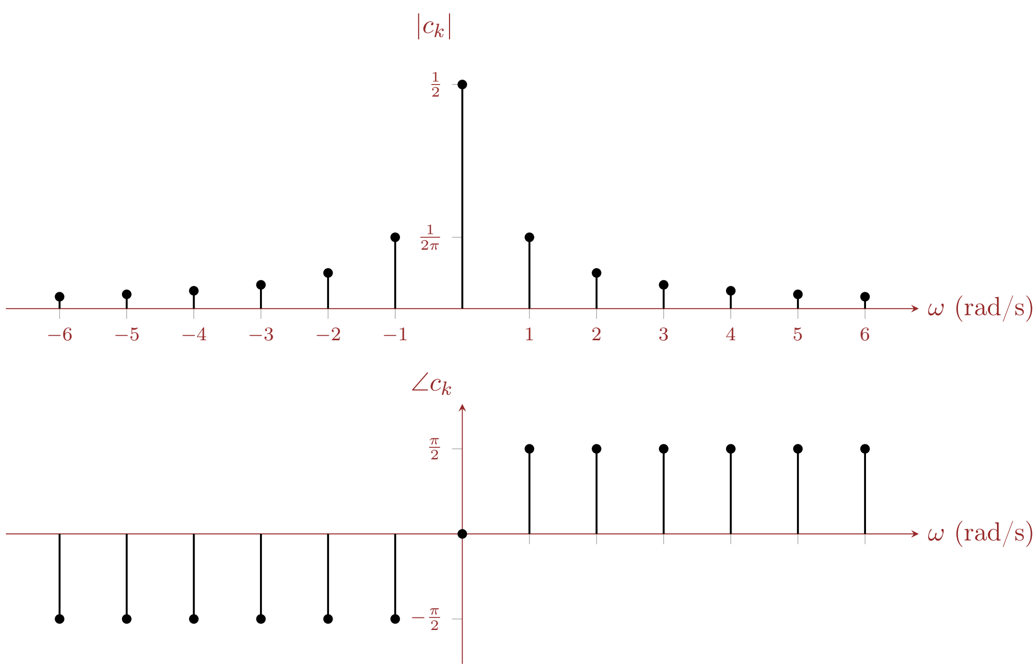


Figure 4: Magnitude and phase spectrum for $|\sin(100t)|$.



2. Fourier series will be

$$f(t) = \sum_k c_k e^{j\omega_0 kt} = \sum_k |c_k| e^{j\angle c_k} e^{j\omega_0 kt}$$

$f(t)$ must be even as $c_{-k} = c_k$ and is therefore made up of terms

$$A\cos(k\omega t) = \frac{A}{2}\left(e^{jk\omega t} + e^{-jk\omega t}\right)$$

Notice:

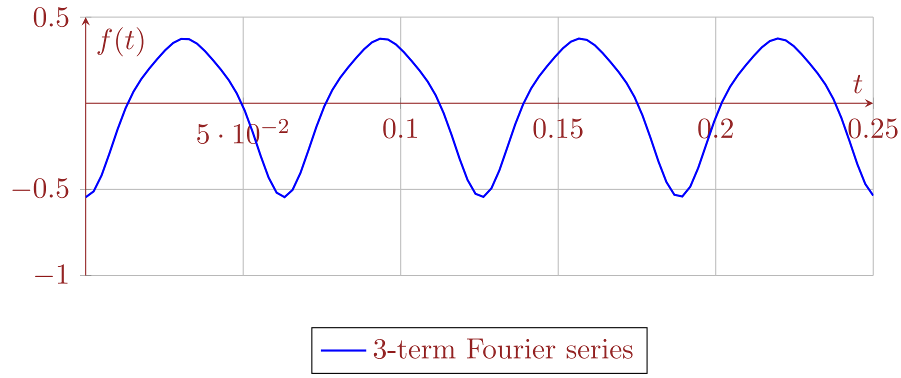
- From the spectrum $|c_k| \neq 0$ only when $\omega = 100n$.
- $|c_0| = 0 \neq 2/\pi$. But

$$|c_k| = \frac{2}{\pi(4k^2-1)} \in \left\{\frac{2}{3\pi}, \cdots, \frac{2}{143\pi}\right\}$$

- Phase $\angle c_k$ implies $c_k < 0$ for all k.

$$f(t) = \sum_{k\neq 0} \frac{-2}{\pi(4k^2-1)} e^{j100kt}$$

Not necessary to answer the questions but plotting the series from $k = -3, \ldots, 3$ we get



3. Each c_k ($k \neq 0$) in the spectrum for $|\sin(100t)|$ has the same magnitude and phase but occurs at $\pm 200k$ instead of $\pm 100k$. So

$$f(t) = \sum_{k\neq 0} \frac{-2e^{j100kt}}{\pi(4k^2-1)} = \sum_{k\neq 0} \frac{-2e^{j(0.5)*200kt}}{\pi(4k^2-1)} = \sum_{k} \frac{-2e^{j(0.5)*200kt}}{\pi(4k^2-1)} - \frac{2}{\pi}$$

i.e. $f(t) = g(0.5t) - 2/\pi = |\sin(50t)| - 2/\pi$

4: Parseval's Theorem

1. Given that the area of a polar function $r(\theta)$ is

$$A = \frac{1}{2}\int_{-\pi}^{\pi} r^2(\theta)\, d\theta$$

Find the area of (a) $f(t) = \cos(t) + \sin(t)$ and (b) $f(t) = 1 - \cos(t)$

⋆ 2. **[Extension]** Using the Fourier expansion of $f(t) = t$ (see the prework questions). Prove that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Parseval's identity for a function $f(t)$ with Fourier series $\sum c_k e^{j\omega kt} = a_0 + \sum a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$.

$$\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\, dt = \sum_{-\infty}^{\infty} |c_k|^2$$

1. **(a)** We have the Fourier series

$$f(t) = \cos(t) + \sin(t) = \left(\frac{1}{2} - \frac{j}{2}\right) e^{jt} + \left(\frac{1}{2} + \frac{j}{2}\right) e^{-jt}$$

So $a_1 = b_1 = 1$ (and $a_k = b_k = 0$ for $k \neq 1$) with $c_1 = (1/2)(1-j)$ and $c_{-1} = (1/2)(1+j)$. Thus

$$\frac{1}{2}\int_{-\pi}^{\pi} |\cos(t) + \sin(t)|^2\, dt = \pi \sum_{-\infty}^{\infty} |c_k|^2 = \pi\left(2\left(\sqrt{1/4 + 1/4}\right)^2\right) = \pi$$

Now $f(\theta) = \cos(\theta) + \sin(\theta) = \sqrt{2}\cos(\theta - \pi/4)$, which is a circle of radius $\sqrt{2}/2$. Since $r(\theta + \pi) = -r(\theta)$ is the same point in polar coordinates, the circle is traced twice over $[-\pi, \pi]$ and the integral is twice the enclosed area, so $A = \pi/2$. Note can use $2c_k = a_k - jb_k$ directly rather than converting to exponential series.

(b) Fourier series for $f(t)$

$$f(t) = 1 - \cos(t) = 1 - \frac{e^{jt}}{2} - \frac{e^{-jt}}{2}$$

So

$$A = \frac{1}{2}\int_{-\pi}^{\pi} |1 - \cos(t)|^2\, dt = \pi\left(1 + \frac{2}{4}\right) = \frac{3}{2}\pi$$

2. We know

$$t = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\sin(kt) = \sum_{k\neq 0} \frac{j(-1)^k}{k} e^{jkt}$$

So

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |t|^2\, dt = \frac{2}{2\pi}\int_0^{\pi} t^2\, dt = \left(\frac{1}{\pi}\right)\left(\frac{\pi^3}{3}\right) = \frac{\pi^2}{3}$$

Applying Parseval

$$\frac{\pi^2}{3} = \sum_{k\neq 0}\left|\frac{(-1)^k}{k}\right|^2 = 2\sum_{k=1}^{\infty}\frac{1}{k^2} \text{ so } \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$

5: Differentiation and Fourier Series

1. **[Extension]** Consider the circuit in Figure 5

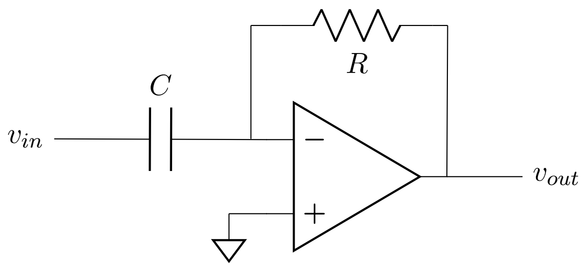


Figure 5: Ideal differentiator

Verify that the input $u(t) = v_{in}(t)$ and output $y(t) = v_{out}(t)$ satisfy

$$y(t) = -RC\frac{d}{dt}u(t)$$

2. Assume $RC = 1$. Suppose $u(t) = u_3(t)$ is the triangular wave given in Figure 1. Verify that $y(t)$ is a square-wave with amplitude 2 and the same frequency as the square-wave given in Figure 2.

3. Consider the Fourier series for a triangular wave $u_3(t)$ and square-wave in Figure 2 i.e.

$$u_3(t) = \sum_k c_k e^{jk\omega t} \text{ and } u_{sq}(t) = \sum_k d_k e^{j\omega kt}$$

   (a) By differentiating term-wise, verify that $u_3'(t) = 2u_{sq}(t)$

   (b) **[Extension]** Find d_k/c_k for each k. How does this compare with the frequency response function (equivalent impedence), $Z_{eq} = -j\omega RC$, of the circuit in Figure 5?

⋆ 4. Find the Fourier series for the signal $u_\delta(t) = \delta(t + \pi/2 - 2k\pi) - \delta(t - \pi/2 - 2k\pi)$ shown in Figure 6 where $k \in \mathbb{Z}$.

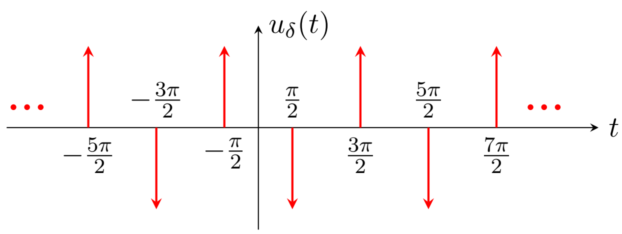


Figure 6: $\delta(t + \pi/2 - 2k\pi) - \delta(t - \pi/2 - 2k\pi)$

1. Assume ideal op-amp so $v_+ = v_- = 0$ i.e. voltages at the inputs are equal and $i_- = 0$ i.e. no current flows into the inputs of the op-amp. Applying KCL at the negative input we have the current through the capacitor is equal to the current through the resistor i.e. $i_C = i_R = i$. Applying the time-domain voltage current relationships for a resistor and capacitor we have

$$C\frac{dV_C}{dt} = C\frac{dv_{in}}{dt} = \frac{V_R}{R} = -\frac{v_{out}}{R}$$

i.e. $v_{out} = -RC\dot{v}_{in}$. Set $y = v_{out}$ and $u = v_{in}$.

2. Notice for $t \in [-\pi/2 + 2\pi n, \pi/2 + 2\pi n]$ that

$$\frac{du_3}{dt} = \frac{\pi + \pi}{\pi} = 2 \text{ so } y(t) = -\frac{du_3}{dt} = -2$$

and $t \in [\pi/2 + 2\pi n, 3\pi/2 + 2\pi n]$

$$\frac{du_3}{dt} = \frac{-\pi - \pi}{\pi} = -2 \text{ so } y(t) = 2$$

So $y(t)$ is a square wave with amplitude 2 and period 2π, inverted relative to $u_{sq}(t)$, i.e. $y(t) = -2u_{sq}(t)$.

3. From Question 1 the Fourier series for the triangular wave is

$$u_3(t) = \sum_{k \text{ odd}} \frac{4}{\pi k^2} e^{jk(t-\pi/2)}$$

And for a square-wave with amplitude 2.

$$2u_{sq}(t) = 2\sum_{k\neq 0} \frac{2}{k\pi} \sin\left(\frac{k\pi}{2}\right) e^{jkt}$$

Differentiating each term in $u_3(t)$ we have

$$\frac{d}{dt}\left(\frac{4e^{jk(t-\pi/2)}}{\pi k^2}\right) = \left(\frac{4e^{-jk\pi/2}}{\pi k^2}\right)\frac{d}{dt}e^{jkt} = \left(\frac{4e^{-jk\pi/2}}{\pi k^2}\right) jke^{jkt} = \left(\frac{4}{\pi k}\right)\sin\left(\frac{k\pi}{2}\right)e^{jkt}$$

Hence term-wise $\dot{u}_3(t) = 2u_{sq}(t)$.

(b) Both c_k and d_k vanish for even k, so the ratio is defined only for odd k, where

$$\frac{d_k}{c_k} = \frac{2\sin(k\pi/2)/(k\pi)}{-4j\sin(k\pi/2)/(\pi k^2)} = \frac{jk}{2}$$

The output is $y(t) = -RC\dot{u}_3(t) = -2u_{sq}(t)$, so the coefficients of $y(t)$ are $-2d_k$ and

$$\frac{-2d_k}{c_k} = -jk = -jk\omega RC$$

with $\omega = 1$ and $RC = 1$. So the circuit multiplies the kth coefficient of the input by Z_{eq} evaluated at the kth harmonic $\omega = k\omega_0$.

4. Notice that $u_\delta(t)$ is the derivative of $u_{sq}(t)/2$. Hence the Fourier series for $u_\delta(t)$ is the derivative of the series of $u_{sq}(t)/2$ i.e.

$$\begin{aligned}
2u_\delta(t) &= \frac{d}{dt}\left(\sum_{k\neq 0} \frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right)e^{jkt}\right) \\
&= \sum_{k\neq 0}\left(\frac{2}{k\pi}\right)\sin\left(\frac{k\pi}{2}\right)(jk)e^{jkt} \\
&= \sum_{k\neq 0}\frac{2j}{\pi}\sin\left(\frac{k\pi}{2}\right)e^{jkt}
\end{aligned}$$

So

$$u_\delta(t) = \frac{j}{\pi}\sum_{k\neq 0}\sin(k\pi/2)e^{jkt}$$

As $u_\delta(t)$ is odd all the c_k are imaginary or zero. Also notice that derivative of an even function is an odd function and derivative of an odd function is even.

$$\text{Even} \longleftarrow \frac{d}{dt} \longrightarrow \text{Odd}$$