Signals and Systems — Tutorial 2

LTI Systems, Impulse Response and Convolution

Tutorial information

- It is expected that you attempt the prework questions before coming to the tutorial each week and submit your *handwritten* solutions to your tutor within the first 15 minutes of your tutorial.
- Not all questions are equally difficult. Difficult questions are marked with a $\star$.
- Questions marked **Extension** go beyond the assessable content for this course and are provided to give a deeper theoretical background and understanding.
- You may not be able to complete all questions within the tutorial time. It is *strongly* recommended that you attempt all questions in your own time and ask your tutor if you are stuck.

Prework questions

A: Reflections and Shifting

1. Given $y(t) = 1 - t$ plot $y(t-2)$ and $y(2-t)$
2. Plot $p_3(t-1)$ and $p_3(2t+1)$, where $p_\tau(t)$ is the pulse with width τ defined in Lecture 2.

B: Sifting

Calculate:

$$\textbf{(a)} \int_0^\infty \delta(t-2)e^{-t}\,\mathrm{d}t, \quad \textbf{(b)} \int_0^4 \delta(t-5)(t^2+1)\,\mathrm{d}t \text{ and } \textbf{(c)} \int_0^\infty \delta(2t-4)t^2\,\mathrm{d}t$$

C: Linearity

Determine if the system $2y(t) = 4\left(u(t-1) + u(t)\right)$ is linear or time-invariant.

Tutorial questions

1: Linearity and Time-invariance

Determine whether each of the systems, with output $y(t)$ and input $u(t)$, are linear or time-invariant.

1. $y(t) = \sin(t)u(t)$
2. $y(t) = t\sin(u(t))$
3. $\dot{y}(t) + 2y(t) = u(t)$ subject to $y(0) = 0$
4. $\dot{y}(t) + 2y(t) = u(t) + 1$ subject to $y(0) = 0$

$\star$ 5. $y(t) = \int_0^t (t-\lambda)u(\lambda)\,\mathrm{d}\lambda$

2: Impulse Response

1. Consider the parallel RC circuit shown in Figure 1 where the input $u(t) = i(t)$ is the current from the source and the output $y(t) = V_C(t)$ is the voltage across the capacitor.

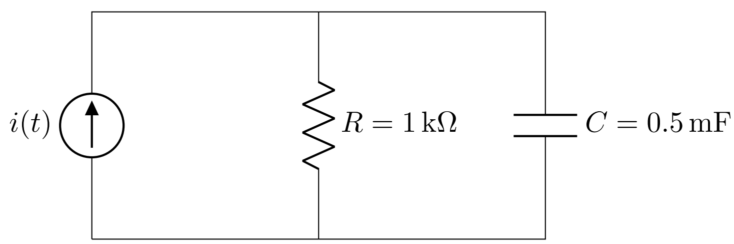

Figure 1: Parallel RC circuit

Show that $u(t)$ and $y(t)$ satisfy

$$Ru(t) = y(t) + RC\dot{y}(t) \tag{1}$$

2. Find the responses of the system
   (a) $y_1(t)$ to the input $u_1(t) = 10\,\delta(t)\,\mathrm{mA}$ and
   (b) $y_2(t)$ to the input $u_2(t) = 10\,H(t)\,\mathrm{mA}$. **Hint:** $y_2(t)$ will be of the form $A(1-e^{-at})H(t)$.

3. How are the solutions $y_1(t)$ and $y_2(t)$ related?

3: Convolution

1. For the following signals $f(t)$ and $g(t)$ find and sketch and calculate the convolution $(f * g)(t)$, specifying the peak value for each of the convolutions.

   (a) $f(t) = \delta(t)$ and $g(t) = e^{-t}H(t)$.

   (b) $f(t) = p_1(t)$ and $g(t) = p_1(t-1)$.

   (c) $f(t) = \sin(\omega t)H(t)$ and $g(t) = H(t) + H(t - \pi/\omega)$.

   $\star$ (d) $f(t) = \omega\cos(\omega t)H(t) + \sin(\omega t)\delta(t)$ and $g(t) = H(t) + H(t - \pi/\omega)$. **Hint:** $f * g' = f' * g$.

2. Consider the responses $y_1(t)$ and $y_2(t)$ from Question 2.

   (a) Show that $(y_1 * H)(t) = y_2(t)$.

   (b) Prove that $y(t) = (y_1 * p_1)(t)$ is the response of (1) to the input $u(t) = 10p_1(t)$.

   (c) Sketch $y(t)$.

4: Projection

Recall that the orthogonal projection of a vector, u, onto a vector, v, is given by

$$proj_v(u) = \left(\frac{\langle u, v\rangle}{\|v\|^2}\right) v$$

1. Let $u = (4, -3)$ and $v = (-2, 3)$ find the projection of u onto v.

2. Let $f(t)$ be the pulse-train in Figure 2. Find the projection of one period of $f(t)$ onto one period of $\sin(t)$ using the formula above and the inner product $\langle \cdot, \cdot \rangle_p$ introduced in Lecture 2.

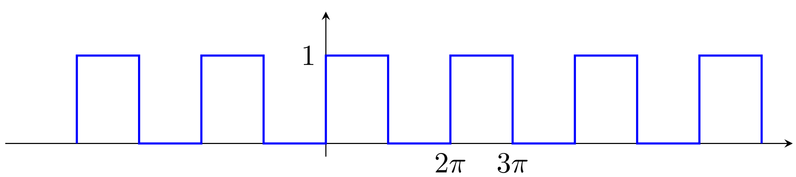

Figure 2: Pulse-train, $f(t)$

3. Project $\sin(2t)$ onto $e^{j\omega t}$. For what values of ω is the projection non-zero?

$\star$ 4. Are $\cos(t)$ and $\sin(t)$ orthogonal? For what values (if any) of ω_c and ω_s are $\cos(\omega_c t)$ and $\sin(\omega_s t)$ orthogonal?

5: Impulse Response of Series Circuits

1. Find the impulse responses $h_R(t)$ and $h_C(t)$ for each of the circuits shown in Figure 3 where the inputs $u_R(t) = i_R(t)$ and $u_C(t) = i_C(t)$ are each of the current sources and the outputs are the voltages across the resistor and capacitor.

Figure 3: Simple series circuits

2. Show the impulse response of the circuit in Figure 4 is $h_R(t) + h_C(t)$, where the output $y(t)$ is the voltage across the pair of components and the input $u(t) = i(t)$. Draw a block diagram for the circuit.

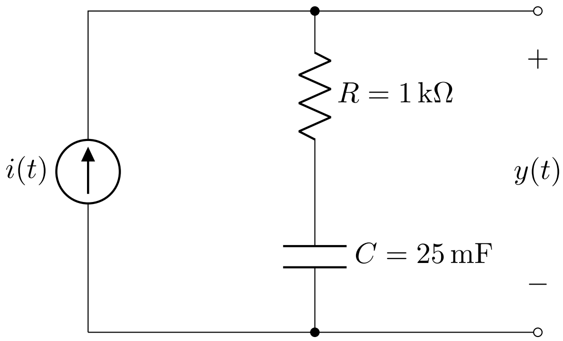

Figure 4: RC-series circuit

3. What is the impulse response of the system shown in Figure 5, where h_A, h_B and h_C are the impulse responses of the LTI systems A, B and C respectively?

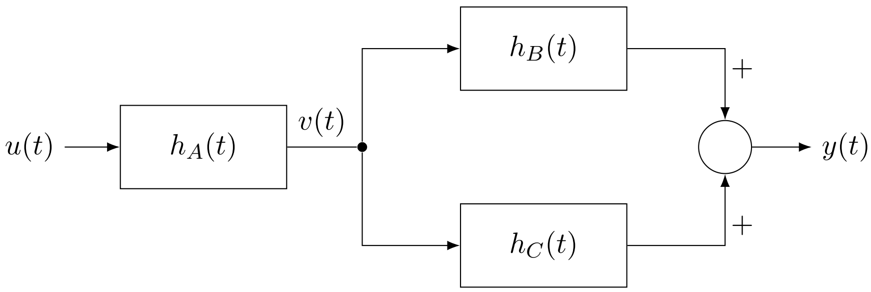

Figure 5: System A cascaded with the parallel sum of systems B and C.

6: Sinusoids and LTI Systems

1. Let $u(t) = e^{j\omega t}$ and $h(t)$ be an integrable[1] complex or real-valued function. Show that

$$y(t) = (h * u)(t) = G(j\omega)\, e^{j\omega t} \qquad \text{where} \quad G(j\omega) := \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, \mathrm{d}\tau.$$

2. Now let $u(t) = \cos(\omega t)$. Using $2\cos(\omega t) = e^{j\omega t} + e^{-j\omega t}$ and assuming $h(t)$ is real, show that

$$y(t) = |G(j\omega)| \cos\left(\omega t + \angle G(j\omega)\right).$$

   What does this say about what an LTI system can and can't do to a sinusoid?

[1] i.e. $h \in L^1$ e.g. there exists $C, \lambda > 0$ such that $|h(t)| < Ce^{-\lambda|t|}$