Signals and Systems — Tutorial 2

LTI Systems, Impulse Response and Convolution

[Tutor version — includes solutions]

Tutorial information

- It is expected that you attempt the prework questions before coming to the tutorial each week and submit your *handwritten* solutions to your tutor within the first 15 minutes of your tutorial.
- Not all questions are equally difficult. Difficult questions are marked with a $\star$.
- Questions marked **Extension** go beyond the assessable content for this course and are provided to give a deeper theoretical background and understanding.
- You may not be able to complete all questions within the tutorial time. It is *strongly* recommended that you attempt all questions in your own time and ask your tutor if you are stuck.

Prework questions

A: Reflections and Shifting

1. Given $y(t) = 1 - t$ plot $y(t-2)$ and $y(2-t)$
2. Plot $p_3(t-1)$ and $p_3(2t+1)$, where $p_\tau(t)$ is the pulse with width τ defined in Lecture 2.

B: Sifting

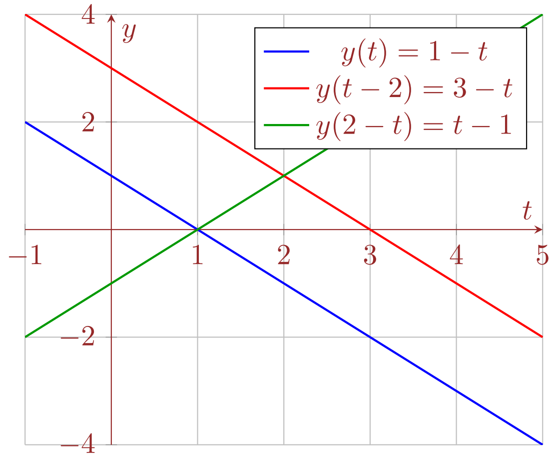
Calculate:

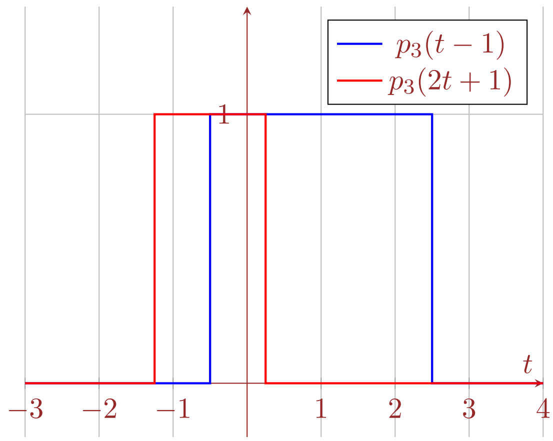
$$\textbf{(a)} \int_0^\infty \delta(t-2)e^{-t}\,\mathrm{d}t, \quad \textbf{(b)} \int_0^4 \delta(t-5)(t^2+1)\,\mathrm{d}t \text{ and } \textbf{(c)} \int_0^\infty \delta(2t-4)t^2\,\mathrm{d}t$$

C: Linearity

Determine if the system $2y(t) = 4\left(u(t-1) + u(t)\right)$ is linear or time-invariant.

A Plots are





B **(a)** e^{-2}, **(b)** 0 and **(c)** 2 (as $\delta(2t-4) = \frac{1}{2}\delta(t-2)$).

C LTI

Tutorial questions

1: Linearity and Time-invariance

Determine whether each of the systems, with output $y(t)$ and input $u(t)$, are linear or time-invariant.

1. $y(t) = \sin(t)u(t)$
2. $y(t) = t\sin(u(t))$
3. $\dot{y}(t) + 2y(t) = u(t)$ subject to $y(0) = 0$
4. $\dot{y}(t) + 2y(t) = u(t) + 1$ subject to $y(0) = 0$

⋆ 5. $y(t) = \int_0^t (t-\lambda)u(\lambda)\,\mathrm{d}\lambda$

1. Linear as

$$y(t) = \sin(t)\left(u_1(t) + bu_2(t)\right) = \sin(t)u_1(t) + b\sin(t)u_2(t) = y_1(t) + by_2(t)$$

Time-varying as the delayed output $y(t+\tau)$ does not equal the output $y(t)$ when the input is delayed.

$$y(t-\tau) = \sin(t-\tau)u(t-\tau) \neq \sin(t)u(t-\tau)$$

Note it is 'time-invariant' if the delay is $\tau = 2\pi k$.

2. Non-linear as

$$y(t) = t\sin(u_1(t) + bu_2(t)) \neq t\sin(u_1(t)) + bt\sin(u_2(t)) = y_1(t) + by_2(t)$$

Time-varying by the same reasoning as Part 1.

3. LTI. Let $y_1(t)$ be the solution when the input is $u_1(t)$ and similarly $y_2(t)$ be the solution when input is $u_2(t)$. Lastly let $y(t)$ be the solution when $u(t) = u_1(t) + bu_2(t)$. We have

$$u_1 + bu_2 = \dot{y}_1 + 2y_1 + b\left(\dot{y}_2 + 2y_2\right) = \frac{d}{dt}\left(y_1 + by_2\right) + 2\left(y_1 + by_2\right) = \dot{y} + 2y = u$$

4. Neither linear nor time-invariant. Linearity:

$$u_1 + bu_2 = \dot{y}_1 + 2y_1 - 1 + b\left(\dot{y}_2 + 2y_2 - 1\right) = \frac{d}{dt}\left(y_1 + by_2\right) + 2\left(y_1 + by_2\right) - 1 - b \neq \dot{y} + 2y - 1 = u.$$

Time invariance: if $u(t) = 0$, delaying the input leaves the response unchanged (the input is still zero) but delaying the output changes the output.

5. Linear as the integral is a linear operator.

$$y(t) = \int (t-\lambda)(u_1(\lambda) + bu_2(\lambda)\,d\lambda = \int (t-\lambda)u_1(\lambda)\,d\lambda + b\int (t-\lambda)u_2(\lambda)\,d\lambda = y_1(t) + by_2(t)$$

Time-varying as delayed output is not equal to response to delayed input i.e. for the *delayed output*

$$y(t-\tau) = \int_0^{t-\tau} (t-\tau-\lambda)u(\lambda)\,d\lambda$$

and for the *delayed input*

$$y(t) = \int_0^{t} (t-\lambda)u(\lambda-\tau)\,d\lambda$$

Change of variables $\lambda' = \lambda - \tau$ gives

$$y(t) = \int_{-\tau}^{t-\tau} (t-\lambda'-\tau)u(\lambda')\,d\lambda'$$

So not time-invariant if $u(t) \neq 0$ for any $t < 0$.

2: Impulse Response

1. Consider the parallel RC circuit shown in Figure 1 where the input $u(t) = i(t)$ is the current from the source and the output $y(t) = V_C(t)$ is the voltage across the capacitor.

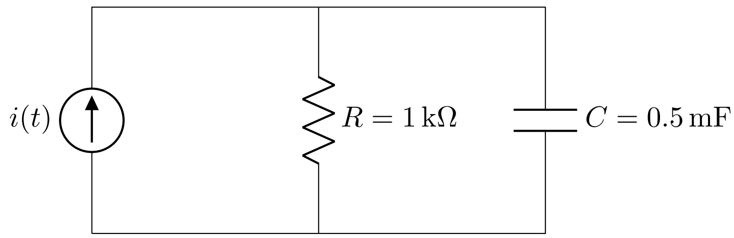


Figure 1: Parallel RC circuit

Show that $u(t)$ and $y(t)$ satisfy

$$Ru(t) = y(t) + RC\dot{y}(t) \tag{1}$$

2. Find the responses of the system
   (a) $y_1(t)$ to the input $u_1(t) = 10\,\delta(t)\,\mathrm{mA}$ and
   (b) $y_2(t)$ to the input $u_2(t) = 10\,H(t)\,\mathrm{mA}$. **Hint:** $y_2(t)$ will be of the form $A(1-e^{-at})H(t)$.
3. How are the solutions $y_1(t)$ and $y_2(t)$ related?

1. KCL implies $u = i = i_R + i_C$. Using $i_R = V_R/R$, $i_C = C\dot{V}_C$ and by KVL that $V_R = V_C = y$. We have

$$u(t) = \frac{y}{R} + C\dot{y} \iff Ru(t) = y + RC\dot{y}$$

2. (a) $y(0) = 0.01/C = 20$. For all $t > 0$ we have the DE

$$RC\dot{y} + y = 0 \iff y(t) = A\exp\left(\frac{-t}{RC}\right)$$

So $y_1(t) = 20e^{-2t}H(t)$.

(b) For all $t \geq 0$. We have the DE $10 = y(t) + RC\dot{y}(t)$. Lot's of ways to solve this. My favourite is

$$\frac{\dot{y}(t)}{y(t) - 10} = \frac{-1}{RC} \iff \ln|y(t) - 10| = \frac{-t}{RC} + K \iff y(t) = Ae^{-t/RC} + 10.$$

Assuming 0 initial conditions, we have $A = -10$. So $y_2(t) = 10(1-e^{-2t})H(t)$.

3. Notice that

$$y_2(t) = \int_0^t y_1(\tau)\,d\tau$$

Linear system so response of $y_2(t) = (u_2 * y_1)(t)$ as $y_1(t) = h(t)$ – the impulse response.

3: Convolution

1. For the following signals $f(t)$ and $g(t)$ find and sketch and calculate the convolution $(f * g)(t)$, specifying the peak value for each of the convolutions.

   (a) $f(t) = \delta(t)$ and $g(t) = e^{-t}H(t)$.

   (b) $f(t) = p_1(t)$ and $g(t) = p_1(t-1)$.

   (c) $f(t) = \sin(\omega t)H(t)$ and $g(t) = H(t) + H(t - \pi/\omega)$.

   $\star$ (d) $f(t) = \omega\cos(\omega t)H(t) + \sin(\omega t)\delta(t)$ and $g(t) = H(t) + H(t - \pi/\omega)$. **Hint:** $f * g' = f' * g$.

2. Consider the responses $y_1(t)$ and $y_2(t)$ from Question 2.

   (a) Show that $(y_1 * H)(t) = y_2(t)$.

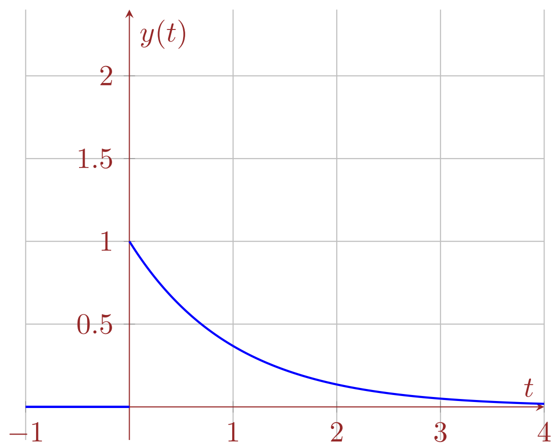
   (b) Prove that $y(t) = (y_1 * p_1)(t)$ is the response of (1) to the input $u(t) = 10p_1(t)$.

   (c) Sketch $y(t)$.

1. Finding the convolutions

   (a) $\delta(t)$ and $e^{-t}H(t)$. **Peak** $= 1$ is at $t = 0$.

$$\int_{-\infty}^{\infty} e^{-\tau}H(\tau)\delta(t-\tau)\,d\tau = e^{-t}H(t)$$



   (b) $p_1(t)$ and $p_1(t-1)$. **Peak** $= 1$ is at $t = 1$.

Method 1 Do this graphically and piecewise. First by the shift property we can consider $(p_1 * p_1)(t)$ and then shift the convolution at the end.

$t < -1$ No intersection so convolution is 0.

$t \in [-1, 0]$ Pulses sliding over each other increasing area so

$$(p_1 * p_1)(t) = \int_{-1}^{t} dt = t + 1$$

$t \in [0, 1]$ Pulses sliding over each other decreasing area so

$$(p_1 * p_1)(t) = \int_{t}^{1} dt = 1 - t$$

$t > 1$ No intersection so convolution is 0.
This gives

$$(p_1 * p_1)(t) = \begin{cases} 0, & |t| > 1 \\ 1 - |t| & |t| \leq 1 \end{cases}$$

Applying the shift we get the convolution $y(t)$ is

$$y(t) = \begin{cases} 0, & |t-1| > 1 \\ 1 - |t-1| & |t-1| \leq 1 \end{cases}$$

Method 2: Or let's have fun... define $f(t) = (p_1 * p_1)(t)$. Defining $\delta_0.5(t) = \delta(t + 0.5)$, we have

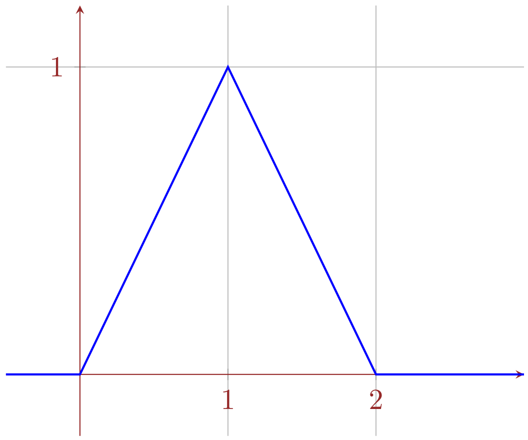
$$\begin{aligned} f'' &= (p_1 * p_1)'' \\ &= (p_1' * p_1)' \\ &= (p_1' * p_1') \\ &= (\delta_{0.5} - \delta_{-0.5}) * (\delta_{0.5} - \delta_{-0.5}) \\ &= \delta_{0.5} * \delta_{0.5} - 2\delta_{0.5} * \delta_{-0.5} + \delta_{-0.5} * \delta_{-0.5} \\ &= \int_{-\infty}^{\infty} \delta_{0.5}(\tau)\delta_{0.5}(t-\tau)\, d\tau - 2\int_{-\infty}^{\infty} \delta_{0.5}(\tau)\delta_{-0.5}(t-\tau)\, d\tau + \int_{-\infty}^{\infty} \delta_{-0.5}(\tau)\delta_{-0.5}(t-\tau)\, d\tau \\ &= \delta_{0.5}(t+0.5) - 2\delta_{0.5}(t-0.5) + \delta_{-0.5}(t-0.5) \\ &= \delta(t+1) - 2\delta(t) + \delta(t-1) \end{aligned}$$

Integrating $f''(t)$ twice we have

$$f(t) = r(t+1) - 2r(t) + r(t-1)$$

where $r(t)$ is the ramp i.e. $r(t) = |t|H(t)$. As we shifted $p_1(t-1)$ by 1 we need to apply a shift to $f(t)$ to get the answer so

$$f(t-1) = r(t) - 2r(t-1) + r(t-2)$$



(c) $\sin(\omega t)H(t)$ and $H(t) + H(t - \pi/\omega)$. **Peak** $= 2/\omega$ for all $t \geq \pi/\omega$.

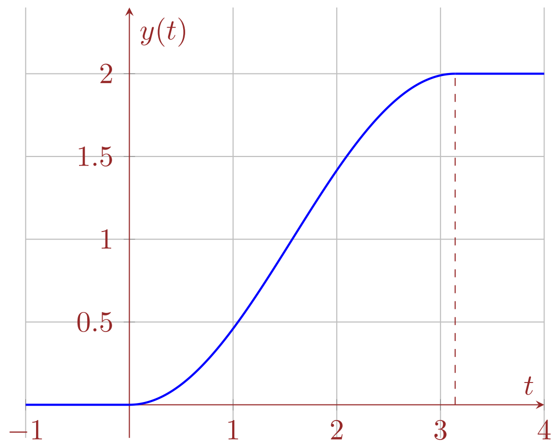
$$\begin{aligned}
(f * g)(t) &= \int_{-\infty}^{\infty} \sin(\omega\tau)H(\tau)\left(H(t-\tau) + H(t - \pi/\omega - \tau)\right) d\tau \\
&= \int_{-\infty}^{\infty} \sin(\omega\tau)H(\tau)H(t-\tau)\,d\tau + \int_{-\infty}^{\infty} \sin(\omega\tau)H(\tau)H\left(t - \frac{\pi}{\omega} - \tau\right) d\tau \\
&= \left(\int_0^t \sin(\omega\tau)\,d\tau\right) H(t) + \left(\int_0^{t-\pi/\omega} \sin(\omega\tau)\,d\tau\right) H(t - \pi/\omega) \\
&= \left(\frac{1-\cos(\omega t)}{\omega}\right) H(t) + \left(\frac{1-\cos(\omega t - \pi)}{\omega}\right) H(t-\pi/\omega) \\
&= \left(\frac{1-\cos(\omega t)}{\omega}\right) H(t) + \left(\frac{1+\cos(\omega t)}{\omega}\right) H(t-\pi/\omega)
\end{aligned}$$

i.e.

$$(f * g)(t) = \begin{cases} 0, & t < 0 \\ \frac{1-\cos(\omega t)}{\omega}, & t \in [0, \pi/\omega] \\ 2/\omega, & t > \pi/\omega \end{cases}$$

Check continuity of convolution: $0 = (1-\cos(0))/\omega$ and $(1-\cos(\omega\pi/\omega))/\omega = 2/\omega$. ☺

Taking $\omega = 1$.



(d) **Peak** $= 1$ at $t = \pi/(2\omega)$. Let $f_c(t) = \sin(\omega t)H(t)$ and $f_d(t) = \omega\cos(\omega t)H(t) + \sin(\omega t)\delta(t))$. Recognise that $f_d(t) = f_c'(t)$ and $g(t)$ in Parts (c) and (d) are the same.

Method 1 Applying the identity $f_d * g = f_c' * g = f_c * g'$ where $g'(t) = \delta(t) + \delta(t - \pi/\omega$. So

$$\begin{aligned}(f_d * g)(t) = (f_c * g') &= \int_{-\infty}^{\infty} \sin(\omega\tau)\left(\delta(t-\tau) + \delta(t - \pi/\omega - \tau)\right) d\tau \\ &= \int_{-\infty}^{\infty} \sin(\omega\tau)\delta(t-\tau)\, d\tau + \int_{-\infty}^{\infty} \sin(\omega\tau)\delta(t - \pi/\omega - \tau)\, d\tau \\ &= \sin(\omega t)H(t) + \sin(\omega t - \pi)H(t - \pi/\omega)\end{aligned}$$

i.e.

$$(f * g)(t) = \begin{cases}\sin(\omega t), & t \in [0, \pi/\omega] \\ 0, & \text{otherwise}\end{cases}$$

Method 2 Now using $(f_d * g) = (f_c * g)'$ we have

$$\begin{aligned}(f_d * g)(t) = (f_c * g)' &= \begin{cases}(d/dt)0, & t < 0 \\ \frac{d}{dt}\left(\frac{1-\cos(\omega t)}{\omega}\right), & t \in [0, \pi/\omega] \\ (d/dt)(2/\omega), & t > \pi/\omega\end{cases} \\ &= \begin{cases}0, & t < 0 \\ \sin(\omega t) & t \in [0, \pi/\omega] \\ 0 & t > \pi/\omega\end{cases} \\ &= \begin{cases}\sin(\omega t), & t \in [0, \pi/\omega] \\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

Method 3 Now doing it the long way, knowing $H(t)H(t) = H(t)$, $\delta(t)H(t) = \delta(t)$

$$(f_d * g)(t) = \int_{-\infty}^{\infty} (\omega\cos(\omega\tau)H(\tau) + \sin(\omega\tau)\delta(\tau))\,(H(t-\tau) + H(t-\pi/\omega-\tau))\;d\tau$$

Solving term by term:

i.

$$\int_{-\infty}^{\infty} \omega\cos(\omega\tau)H(\tau)H(t-\tau) = \omega\int_0^t \cos(\omega\tau) = \sin(\omega t)H(t)$$

,

ii.

$$\begin{aligned}\int_{-\infty}^{\infty} \omega\cos(\omega\tau)H(\tau)H\left(t-\frac{\pi}{\omega}-\tau\right)\,d\tau &= \omega\int_0^{t-\pi/\omega} \cos(\omega\tau)\,d\tau \\ &= \sin(\omega t-\pi)H\left(t-\frac{\pi}{\omega}\right)\end{aligned}$$

,

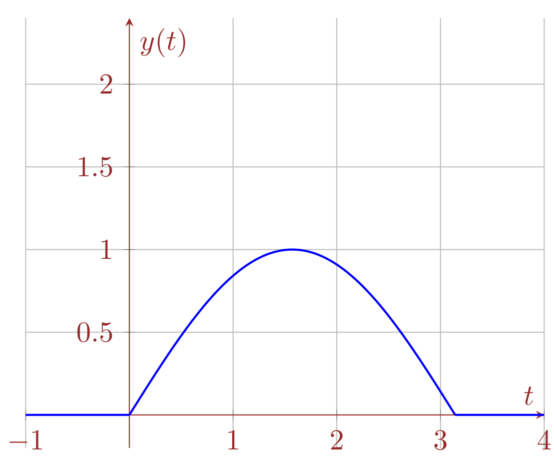
iii.

$$\int_{-\infty}^{\infty} \sin(\omega\tau)\delta(\tau)H(t-\tau)\,d\tau = \int_{-\infty}^{\infty} \sin(\omega\tau)\delta(\tau)H(t-\pi/\omega-\tau)\,d\tau = 0$$

So

$$(f_d * g)(t) = \sin(\omega t)H(t) - \sin(\omega t)H(t-\pi/\omega) = \begin{cases}\sin(\omega t), & [0,\pi/\omega] \\ 0, & \text{otherwise}\end{cases}$$

Taking $\omega = 1$.



2. $y_1 = 20e^{-2t} = Ae^{-at}$ and $y_2(t) = 10(1-e^{-2t})H(t) = (A/a)(1-e^{-at})H(t)$.

(a) y_1 with H

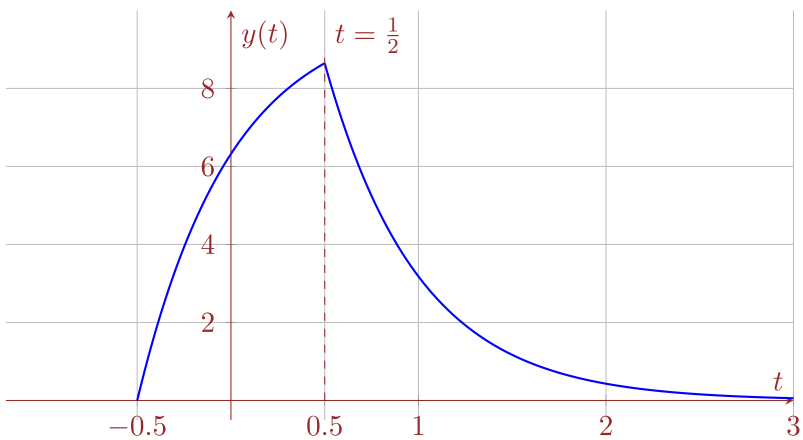
$$
\begin{aligned}
(y_1 * H)(t) &= \int_{-\infty}^{\infty} y_1(t-\tau)H(\tau)\,d\tau = A\int_0^{\infty} e^{-a(t-\tau)}H(t-\tau)\,d\tau = Ae^{-at}\int_0^t e^{-a\tau}\,d\tau \\
&= \frac{A}{a}\left(1-e^{-at}\right)H(t) \\
&= 10\left(1-e^{-2t}\right)H(t) \\
&= y_2(t)
\end{aligned}
$$

(b) Need to find solution to (1) with $u = 10p_1$ and convolve y_1 with p_1 or substitute $y_1 * p_1$ into (1) and show LHS = RHS. The second method is *much better!* Want to show

$$
\begin{aligned}
Ru &= y + RC\dot{y} \\
\iff R10p_1 &= y_1 * p_1 + RC(y_1 * p_1)' \\
\iff 10Rp_1 &= y_1 * p_1 + RC(y_1' * p_1) \\
\iff 10Rp_1 &= (y_1 + RCy_1') * p_1 \\
\iff 10Rp_1 &= (R10\delta) * p_1 \\
\iff p_1 &= \delta * p_1 = p_1
\end{aligned}
$$

(c) Need to find

$$
y(t) = \begin{cases} 10(1-e^{-2t-1}), & t \in [-1/2, 1/2] \\ 10(e^1 - e^{-1})e^{-2t}), & t > 1/2 \end{cases}
$$



Find the peak values.

4: Projection

Recall that the orthogonal projection of a vector, u, onto a vector, v, is given by

$$
proj_v(u) = \left(\frac{\langle u, v\rangle}{\|v\|^2}\right) v
$$

1. Let $u = (4, -3)$ and $v = (-2, 3)$ find the projection of u onto v.

2. Let $f(t)$ be the pulse-train in Figure 2. Find the projection of one period of $f(t)$ onto one period of $\sin(t)$ using the formula above and the inner product $\langle \cdot, \cdot \rangle_p$ introduced in Lecture 2.

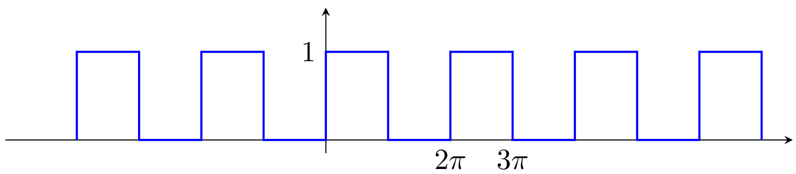


Figure 2: Pulse-train, $f(t)$

⋆ 3. Project $\sin(2t)$ onto e^{jkt}. For what values of k is the projection non-zero?

⋆ 4. Are $\cos(t)$ and $\sin(t)$ orthogonal? For what values (if any) of ω_c and ω_s are $\cos(\omega_c t)$ and $\sin(\omega_s t)$ orthogonal?

The motivation for this question is the Fourier series and transform which are projections of functions onto an orthonormal basis for L^2.

1. Vector projection

$$\begin{aligned} proj_u(v) &= \left(\frac{\langle (4,-3),(-2,3)\rangle}{\|u\|^2}\right)(-2,3) \\ &= \frac{4(-2)+3(-3)}{2^2+3^2}(-2,3) \\ &= \frac{-17}{13}(-2,3) \end{aligned}$$

2. Recall

$$\langle f, g\rangle_p = \frac{1}{T}\int_0^T f(\tau)\overline{g(\tau)}\,d\tau$$

Period of $\sin(t)$ and $f(t)$ is $T = 2\pi$. So

$$\langle f, g\rangle_p = \frac{1}{T}\int_0^T f(\tau)\overline{g(\tau)}\,d\tau = \frac{1}{2\pi}\int_0^{2\pi} \sin(\tau)f(\tau)\,d\tau = \frac{1}{2\pi}\int_0^{\pi}\sin(\tau)\,d\tau$$

Hence

$$\langle \sin(t), f(t)\rangle_P = \frac{1}{2\pi}\left(\cos(0) - \cos(\pi)\right) = \frac{1}{2\pi}$$

Recall $\|\cdot\|^2 = |\langle \cdot, \cdot\rangle|$ for any norm and inner-product. So

$$\|\sin(t)\|_P^2 = \langle \sin(t), \sin(t)\rangle_P = \frac{1}{2\pi}\int_0^{2\pi}\sin^2(\tau)\,d\tau = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-\cos(2\tau)}{2}\,d\tau = \frac{1}{2}$$

So

$$proj_{\sin(t)}(f(t)) = \left(\frac{\langle f(t), \sin(t)\rangle}{\|\sin(t)\|^2}\right)\sin(t) = \frac{2\sin(t)}{\pi}$$

NB projecting $\sin(t)$ onto $f(t)$ is slightly easier.

3. Recall
$$\text{proj}_{(e^{jkt})}(\sin(2t)) = \left(\frac{\langle e^{jkt}, \sin(2t)\rangle_p}{\|e^{jkt}\|^2}\right) e^{jkt}.$$

Using $\sin(2t) = (e^{j2t} - e^{-j2t})/2j$ (from Euler's formula), the inner product is

$$\begin{aligned}
\langle e^{jkt}, \sin(2t)\rangle_p = \frac{1}{2\pi}\int_0^{2\pi} \sin(2t)\overline{e^{jkt}}\, dt &= \frac{1}{2\pi}\int_0^{2\pi} \sin(2t)e^{-jkt}\, dt \\
&= \frac{1}{4\pi j}\int_0^{2\pi} \left(e^{j(2-k)t} - e^{-j(2+k)t}\right) dt \\
&= \frac{1}{4\pi j}\int_0^{2\pi} \left(e^{jat} - e^{jbt}\right) dt \\
&= \frac{1}{4\pi j}\left(\frac{e^{j2\pi a} - 1}{ja} - \frac{e^{j2\pi b} - 1}{jb}\right) \\
&= \frac{1}{4\pi}\left(\frac{1 - e^{j2\pi a}}{a} - \frac{1 - e^{j2\pi b}}{b}\right)
\end{aligned}$$

where $a = 2 - k$ and $b = 2 + k$. As $\|e^{jkt}\| = 1$ for all k and for all t, we have

$$\text{proj}_{(e^{jkt})}(\sin(2t)) = \frac{1}{4\pi}\left(\frac{1 - e^{j2\pi a}}{a} - \frac{1 - e^{j2\pi b}}{b}\right) e^{jkt}$$

k for which proj $\neq$ 0: Define

$$c_k = \frac{1}{4\pi}\left(\frac{1 - e^{j2\pi a}}{a}\right) \text{ and } c_{-k} = \frac{1}{4\pi}\left(\frac{1 - e^{j2\pi b}}{b}\right)$$

Notice for all $k \neq 2$ that $c_k = 0$ and for all $k \neq -2$ that $c_{-k} = 0$, as $e^{2\pi jk} = 1$ for any integer k. Thus the projection is 0 for all $k \neq \pm 2$.

Extension

Whilst not necessary for the question let's find the projection when $k = \pm 2$. The only problem is that either a or $b = 0$ when $k = \pm 2$. Let's consider c_2. From lectures
$$e^{2\pi ja} = 1 + 2\pi ja + \frac{(2\pi ja)^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{(2\pi ja)^n}{n!}$$

So

$$
\begin{aligned}
c_2 = \lim_{a\to 0}\left(\frac{1-e^{2\pi ja}}{4\pi a}\right) &= \left(\frac{1}{4\pi}\right)\lim_{a\to 0}\left(\frac{1}{a}\left(1-\sum_{n=0}^{\infty}\frac{(2\pi ja)^n}{n!}\right)\right)\\
&= \left(\frac{1}{4\pi}\right)\lim_{a\to 0}\left(-\sum_{n=1}^{\infty}\frac{(2\pi ja)^n}{a(n!)}\right)\\
&= \left(\frac{-1}{4\pi}\right)\lim_{a\to 0}\left(\sum_{n=1}^{\infty}\frac{(2\pi j)^n a^{n-1}}{n!}\right)\\
&= \left(\frac{-1}{4\pi}\right)\left(\sum_{n=1}^{\infty}\frac{(2\pi j)^n}{n!}\left(\lim_{a\to 0}a^{n-1}\right)\right)\\
&= \frac{-2\pi j}{4\pi}\\
&= \frac{-j}{2}
\end{aligned}
$$

Similarly $c_{-2} = j/2$. So

$$
\text{proj}_{(e^{jkt})}\big(\sin(2t)\big) = c_k e^{jkt} \text{ where } c_k = \begin{cases}\pm\frac{j}{2}, & k=\mp 2,\\ 0, & \text{otherwise.}\end{cases}
$$

which was *a lot* of work for essentially Euler's formula!

4. To check if $\sin(t)$ and $\cos(t)$ are orthogonal. Need to find the inner product:

$$
\langle \sin(t), \cos(t)\rangle_p = \frac{1}{2\pi}\int_0^{\infty}\sin(t)\overline{\cos(t)}\,dt = \frac{1}{2\pi}\int_0^{\infty}\sin(t)\cos(t)\,dt
$$

Either integrate by parts twice or using $\sin(t)\cos(t) = \sin(2t)$ or

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{\infty}\sin(t)\cos(t)\,dt &= \frac{1}{8\pi j}\int_0^{2\pi}\left(e^{jt}-e^{-jt}\right)\left(e^{jt}-e^{-jt}\right)\,dt\\
&= \frac{1}{8\pi j}\int_0^{2\pi} e^{j2t}-e^{-j2t}\,dt\\
&= \frac{-1}{16\pi}\left(e^{j2t}+e^{-j2t}\right|_0^{2\pi}\\
&= 0
\end{aligned}
$$

As the inner product is 0 the projection is 0. Just expecting to notice that they are orthogonal for all integer ω_c and ω_s.

5: Impulse Response of Series Circuits

1. Find the impulse responses $h_R(t)$ and $h_C(t)$ for each of the circuits shown in Figure 3 where the inputs $u_R(t) = i_R(t)$ and $u_C(t) = i_C(t)$ are each of the current sources and the outputs are the voltages across the resistor and capacitor.

Figure 3: Simple series circuits

2. Show the impulse response of the circuit in Figure 4 is $h_R(t) + h_C(t)$, where the output $y(t)$ is the voltage across the pair of components and the input $u(t) = i(t)$. Draw a block diagram for the circuit.

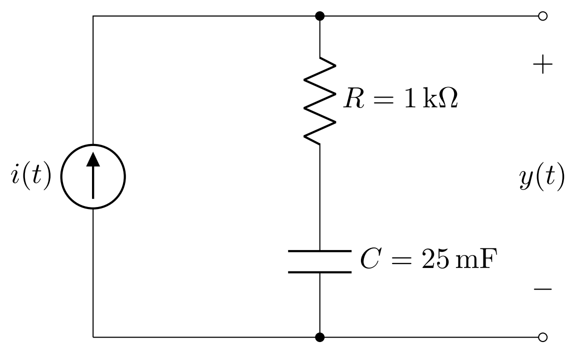


Figure 4: RC-series circuit

3. What is the impulse response of the system shown in Figure 5, where h_A, h_B and h_C are the impulse responses of the LTI systems A, B and C respectively?

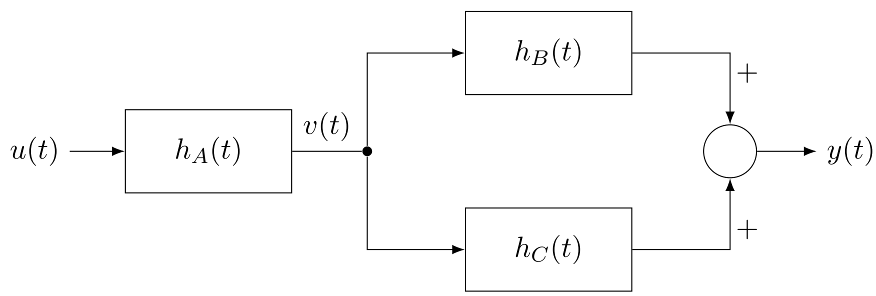


Figure 5: System A cascaded with the parallel sum of systems B and C.

1. From the V-I relationships for a resistor and capacitor we have

   $y_R(t) = Ru_R(t) = 1000u_R(t) = 1000\delta(t)$ and $\dot{y}_C(t) = u_C(t)/C = 40u_C(t) = 40\delta(t)$

   So
   $$h_R(t) = \begin{cases} 1000, & t = 0 \\ 0, & \text{otherwise} \end{cases}$$

   and $h_C(t) = H(t)/C = 40H(t)$ assuming $y_C(0^-) = 0$ which we know as $u(t) = \delta(t)$.

2. Considering the circuit in Figure 4, by KVL
   $$y(t) = V_R + V_C = i(t)R + \frac{1}{C}\int_0^t i(\tau)\,d\tau = \delta R + H(t)/C = h_R(t) + h_C(t)$$

3. $y(t)$ will be the sum of the outputs of systems B and C which respond to the output $v(t)$ of system A. So output will be $y(t) = h_A(t) * (h_B(t) + h_C(t))$.

6: Sinusoids and LTI Systems

1. Let $u(t) = e^{j\omega t}$ and $h(t)$ be an integrable[1] complex or real-valued function. Show that

$$y(t) = (h * u)(t) = G(j\omega)\, e^{j\omega t} \qquad \text{where} \quad G(j\omega) := \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, \mathrm{d}\tau.$$

2. Now let $u(t) = \cos(\omega t)$. Using $2\cos(\omega t) = e^{j\omega t} + e^{-j\omega t}$ and assuming $h(t)$ is real, show that

$$y(t) = |G(j\omega)| \cos\left(\omega t + \angle G(j\omega)\right).$$

What does this say about what an LTI system can and can't do to a sinusoid?

1. Taking the convolution

$$y(t) = (h * u)(t) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)}\, d\tau = e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau}\, d\tau = e^{j\omega t} H(j\omega)$$

2. Convert to exponential form i.e. $H(j\omega) = |H(j\omega)| e^{j\angle H(j\omega)}$.

$$y(t) = (h * u)(t) = |H(j\omega)| e^{j\angle H(j\omega)} \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right) = |H(j\omega)| \cos(\omega t + \angle H(j\omega))$$

Assuming h is the impulse response for an LTI system this means $h * u$ is the response of the system to any input u. So LTI systems can alter amplitude and shift phase but cannot change frequency. So response of a LTI system to a sinusoidal input will be a sinusoid with some phase shift and gain.

[1] i.e. $h \in L^1$ e.g. there exists $C, \lambda > 0$ such that $|h(t)| < Ce^{-\lambda|t|}$