Signals and Systems — Tutorial 1

Complex Numbers and Introduction to Signal Norms

Tutorial information

- It is expected that you attempt the prework questions before coming to the tutorial each week and submit your *handwritten* solutions to your tutor within the first 15 minutes of your tutorial.
- Not all questions are equally difficult. Difficult questions are marked with a $\star$.
- Questions marked **Extension** go beyond the assessable content for this course and are provided to give a deeper theoretical background and understanding.
- You may not be able to complete all questions within the tutorial time. It is *strongly* recommended that you attempt all questions in your own time and ask your tutor if you are stuck.

Prework questions

A: Polar form

Express the following numbers in polar form

1. $3 + 4j$
2. $(1 - 3j)(1 + 3j)$
3. $e^{(j\pi)/2} + 1$
4. $(2 + 2j)^3$
5. $(1 + j)/(-4 + 3j)$

B: Cartesian form

Express the following numbers in Cartesian form

1. $3e^{\frac{j\pi}{4}}$
2. $1/(e^{j\pi/4} + 1)$
3. $e^{\frac{j\pi}{4}} + 2e^{-\frac{j\pi}{4}}$

C: Differential Equations

Solve $\ddot{x} + 2\dot{x} + 2x = \sin(t)$, when $x(0) = 1$ and $\dot{x}(0) = 0$

Tutorial questions

1: Euler's Formula

1. Show that $e^{a+jb} + e^{a-jb} = 2e^a \cos(b)$.
2. Prove $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $A\cos(bt) + B\sin(bt) = C\cos(bt + \phi)$.
3. Suppose $x_1(t) = 6\sqrt{2}\cos(\omega t + \pi/4)$ and $x_2 = 2\cos(\omega t)$. Using Euler's formula find:
   (a) $x_1 + x_2$,
   (b) $x_1 \times x_2$
   (c) x_1/x_2.

2: Complex Valued Functions

Consider the complex-valued function, $f(\omega)$, of a real variable ω

$$f(\omega) = \frac{1 + j\omega}{3 - 2j\omega}$$

find the real and imaginary parts, magnitude and phase (angle) of f in terms of ω.

3: Differential Equations

Prove that the complex-valued function $x(t) = A\exp(zt) + B$, where $A, B \in \mathbb{R}$ and $z \in \mathbb{C}$, is a solution the following differential equations and plot the trajectory of $x(t)$ in the complex plane.

1. $\dot{x} + 2x = 4$ subject to $x(0) = 1$
2. $4\ddot{x} - 2\dot{x} + (5/2)x = 0$ subject to $x(0) = 1$
3. $\ddot{x} + 2\dot{x} + 5x = 5$ subject to $x(0) = -1$

4: Application – RLC Circuits

Consider the RLC series circuit shown in Figure 1 where the input $u(t) = v(t)$ is the voltage of the supply and the output $y(t)$ is the voltage across the capacitor $y(t) = V_C(t)$ with $y(0) = 10\,\mathrm{V}$ and $i(0) = -5\,\mathrm{A}$.

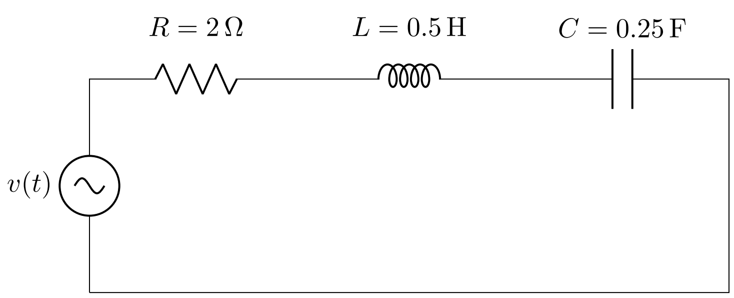


Figure 1: RLC series circuit

1. Show the output satisfies
$$LC\ddot{y}(t) + RC\dot{y}(t) + y(t) = u(t) \tag{1}$$

2. Let
$$z = \left(-R \pm \sqrt{R^2 - 4L/C}\right)/(2L)$$
and $a = Re(z)$ and $b = Im(z)$. Suppose $u(t) = 0$. Prove that $y(t) = A\exp((a \pm jb)t)$ are solutions to (1) for all $t \geq 0$.

3. Plot the trajectory of $y(t)$ in the complex plane. What would happen to this trajectory if $R = 0$ or $R < 0$.

4. Hence, prove that $y(t) = Be^{at}\cos(bt + \phi)$ is a solution to (1). **Hint:** if $y_1(t)$ and $y_2(t)$ are solutions to (1) then $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution.

5. Find and plot the current $i(t)$ as a function of t.

5: Energy and Power

1. The periodic signal $f(t)$ in Figure 2 is an idealised version of the pulse train emitted by the Kurnell weather radar. Calculate its energy, power and RMS.

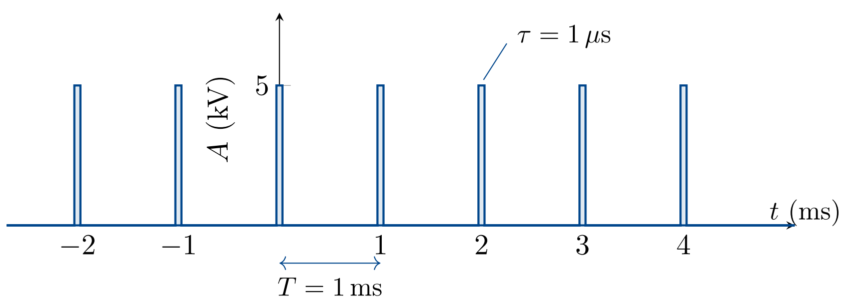


Figure 2: Pulse-train emitted by the Kurnell weather radar, $f(t)$ – excluding the carrier frequency to be covered in Week 11.

2. Find the energy and power of the functions
   (a) $f(t) + 1$
   (b) $f(t + \tau)$ for $\tau > 0$
   (c) **[Extension]** $e^{-|t|} f(t)$

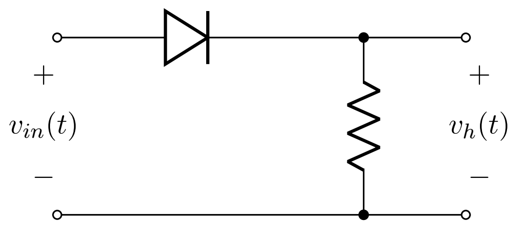
3. What happens to the energy of a periodic signal if the power is non-zero?

4. Find the energy, power and RMS of and sketch the output, $y(t)$:

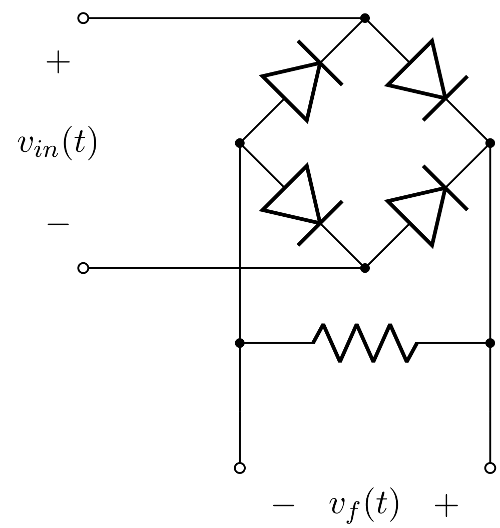
   (a) of the half-wave rectifier circuit in Figure 3a, i.e. $y(t) = v_h(t)$

   (b) of the full-wave rectifier in Figure 3b, i.e. $y(t) = v_f(t)$.

   Assume the input $u(t) = v_{in}(t) = \sin(100\pi t)$ for both circuits and all diodes are ideal.



(a) Half-wave rectifier



(b) full-wave rectifier

⋆ 5. Find the energy, power and RMS for $v_h(t)$ and $v_f(t)$ above but assuming that the diodes have a threshold voltage of 0.7 V.

6: Dirac δ-function

1. For any $\varepsilon > 0$ find the energy and power of the signal

$$f(\varepsilon, t) = \begin{cases} 1/\varepsilon, & |t| \leq \varepsilon/2 \\ 0, & \text{otherwise} \end{cases}$$

   what happens to the energy and power as $\varepsilon \to 0$?

⋆ 2. Use $f(\varepsilon, t)$ to prove that $\delta(\alpha t) = \delta(t)/|\alpha|$.