Signals and Systems — Tutorial 1

Complex Numbers and Introduction to Signal Norms

[Tutor version — includes solutions]

Tutorial information

- It is expected that you attempt the prework questions before coming to the tutorial each week and submit your *handwritten* solutions to your tutor within the first 15 minutes of your tutorial.
- Not all questions are equally difficult. Difficult questions are marked with a $\star$.
- Questions marked **Extension** go beyond the assessable content for this course and are provided to give a deeper theoretical background and understanding.
- You may not be able to complete all questions within the tutorial time. It is *strongly* recommended that you attempt all questions in your own time and ask your tutor if you are stuck.

Prework questions

A: Polar form

Express the following numbers in polar form

1. $3 + 4j$
2. $(1 - 3j)(1 + 3j)$
3. $e^{(j\pi)/2} + 1$
4. $(2 + 2j)^3$
5. $(1 + j)/(-4 + 3j)$

B: Cartesian form

Express the following numbers in Cartesian form

1. $3e^{\frac{j\pi}{4}}$
2. $1/(e^{j\pi/4} + 1)$
3. $e^{\frac{j\pi}{4}} + 2e^{-\frac{j\pi}{4}}$

C: Differential Equations

Solve $\ddot{x} + 2\dot{x} + 2x = \sin(t)$, when $x(0) = 1$ and $\dot{x}(0) = 0$

QA

1. $5e^{j0.9273}$
2. 10
3. $\sqrt{2}\,e^{j\pi/4}$
4. $16\sqrt{2}\,e^{j3\pi/4}$
5. $\dfrac{\sqrt{2}}{5}\,e^{-j1.7127}$

QB

1. $\dfrac{3}{\sqrt{2}} + j\,\dfrac{3}{\sqrt{2}}$
2. $\dfrac{1}{2} - j\,\dfrac{\sqrt{2}-1}{2}$
3. $\dfrac{3}{\sqrt{2}} - j\,\dfrac{1}{\sqrt{2}}$

QC

- $x_h(t) = e^{-t}\left(\frac{7}{5}\cos(t) + \frac{6}{5}\sin(t)\right)$
- $x_p(t) = -\frac{2}{5}\cos(t) + \frac{1}{5}\sin(t)$
- $x(t) = e^{-t}\left(\frac{7}{5}\cos(t) + \frac{6}{5}\sin(t)\right) - \frac{2}{5}\cos(t) + \frac{1}{5}\sin(t)$

Tutorial questions

1: Euler's Formula

1. Show that $e^{a+jb} + e^{a-jb} = 2e^a \cos(b)$.
2. Prove $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $A\cos(bt) + B\sin(bt) = C\cos(bt+\phi)$.
3. Suppose $x_1(t) = 6\sqrt{2}\cos(\omega t + \pi/4)$ and $x_2 = 2\cos(\omega t)$. Using Euler's formula find:
   (a) $x_1 + x_2$,
   (b) $x_1 \times x_2$
   (c) x_1/x_2.

1. Use Euler's formula and $\cos(-b) = \cos(b)$ and $\sin(-b) = -\sin(b)$.

$$\begin{aligned} e^{a+jb} + e^{a-jb} &= e^a \left(e^{jb} + e^{-jb}\right) \\ &= e^a \left(\cos(b) + j\sin(b) + \cos(-b) + j\sin(-b)\right) \\ &= 2e^a \cos(b) \end{aligned}$$

2. Prove first identity using Euler then use it to prove second.

$$\begin{aligned} \cos(x+y) + j\sin(x+y) &= e^{j(x+y)} \\ &= e^{jx}e^{jy} \\ &= (\cos(x) + j\sin(x))(\cos(y) + j\sin(y)) \\ &= \cos(x)\cos(y) + j(\cos(x)\sin(y) + \cos(y)\sin(x)) - \sin(x)\sin(y) \end{aligned}$$

Result follows by equating real and imaginary parts which also proves the identity $\cos(x)\sin(y) + \cos(y)\sin(x) = \sin(x+y)$. Using above

$$C\cos(bt+\phi) = C\cos(bt)\cos(\phi) - C\sin(bt)\sin(\phi)$$

Result follows setting $A = C\cos(\phi)$ and $B = -C\sin(\phi)$.

3. (a) $x_1 = Re(6\sqrt{2}e^{j(\omega t+\pi/4)})$ and $x_2 = Re(2e^{j\omega t})$. For $z_1, z_2 \in \mathbb{C}$ we have $Re(z_1) + Re(z_2) = Re(z_1 + z_2)$. So

$$\begin{aligned} 6\sqrt{2}e^{j(\omega t+\pi/4)} + 2e^{j\omega t} &= e^{j\omega t}(6\sqrt{2}e^{j\pi/4} + 2) \\ &= e^{j\omega t}(6\sqrt{2}\cos(\pi/4) + 6\sqrt{2}j\sin(\pi/4) + 2) \\ &= e^{j\omega t}(8 + 6j) \\ &= 10e^{j(\omega t + \arctan(3/4))} \end{aligned}$$

Taking the real part we have $x_1 + x_2 = 10\cos(\omega t + 0.64)$.

(b) For multiplication and division $Re(z_1)Re(z_2) \neq Re(z_1 z_2)$ and $Re(z_1)/Re(z_2) \neq Re(z_1/z_2)$ in general. Use Part 1. Define $b_1 = \omega t + \pi/4$ and $b_2 = \omega t$.

$$\begin{aligned} x_1 x_2 &= (3\sqrt{2}(e^{jb_1} + e^{-jb_1}))(e^{jb_2} + e^{-jb_2}) \\ &= 3\sqrt{2}\left(e^{j(b_1+b_2)} + e^{j(b_1-b_2)} + e^{j(-b_1+b_2)} + e^{j(-b_1-b_2)}\right) \end{aligned}$$

As $b_1 + b_2 = 2\omega t + \pi/4 = -(-b_1 - b_2)$ and $b_1 - b_2 = \pi/4 = -(-b_1 + b_2)$ we have

$$x_1 x_2 = 3\sqrt{2}(2\cos(2\omega t + \pi/4) + 2\cos(\pi/4)) = 6\sqrt{2}\cos(2\omega t + \pi/4) + 6$$

(c) Use the identity derived in Part 2 (derived using Euler's formula so technically still using Euler's formula).

$$\begin{aligned} \frac{x_1}{x_2} &= 3\sqrt{2}\left(\frac{\cos(\omega t + \pi/4)}{\cos(\omega t)}\right) \\ &= 3\left(\frac{\cos(\omega t) - \sin(\omega t)}{\cos(\omega t)}\right) \\ &= 3\left(1 - \tan(\omega t)\right). \end{aligned}$$

2: Complex Valued Functions

Consider the complex-valued function, $f(\omega)$, of a real variable ω

$$f(\omega) = \frac{1 + j\omega}{3 - 2j\omega}$$

find the real and imaginary parts, magnitude and phase (angle) of f in terms of ω.

To find real and imaginary parts multiply by the complex conjugate:

$$\frac{1 + j\omega}{3 - 2j\omega}\frac{3 + 2j\omega}{3 + 2j\omega} = \frac{3 - 2\omega^2}{9 + 4\omega^2} + j\frac{5\omega}{9 + 4\omega^2}$$

Magnitude, important is $|num/den| = |num|/|den|$ and magnitude of complex numbers is just Pythagoras' theorem. Point out they have already found squared magnitude of the denominator when converting to Cartesian form.

$$\left|\frac{1 + j\omega}{3 - 2j\omega}\right| = \frac{|1 + j\omega|}{|3 - 2j\omega|} = \frac{\sqrt{1 + \omega^2}}{\sqrt{9 + 4\omega^2}} = \sqrt{\frac{1 + \omega^2}{9 + 4\omega^2}}$$

Phase, important that $\angle(num/den) = \angle num - \angle den$ which just comes from exponent rules when dividing exponents. Also point out it is just $\tan = opp/adj$ of a right-angled triangle. Two options either use real and imaginary parts above or do $\angle num - \angle den$.

$$\angle f(\omega) = \arctan(\omega) - \arctan\left(-2\omega/3\right) = \arctan(\omega) + \arctan\left(2\omega/3\right).$$

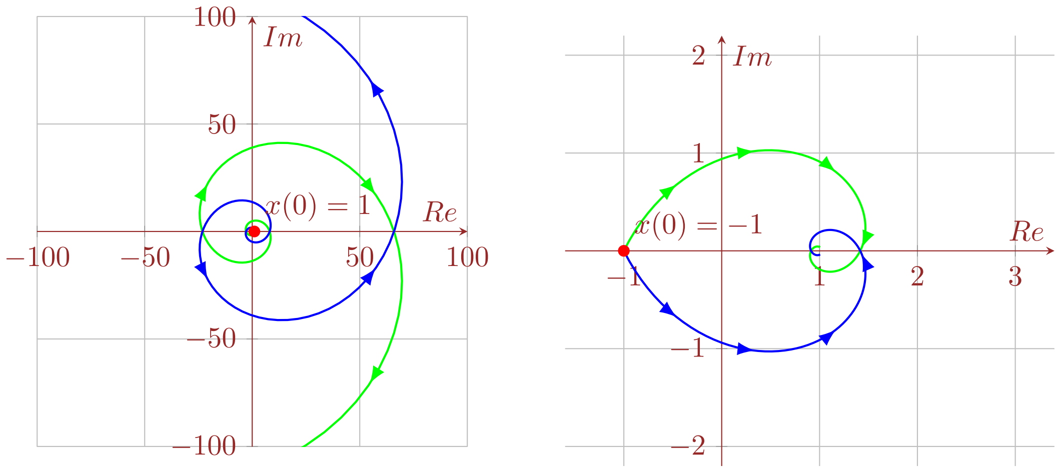
3: Differential Equations

Prove that the complex-valued function $x(t) = A\exp(zt) + B$, where $A, B \in \mathbb{R}$ and $z \in \mathbb{C}$, is a solution the following differential equations and plot the trajectory of $x(t)$ in the complex plane.

1. $\dot{x} + 2x = 4$ subject to $x(0) = 1$
2. $4\ddot{x} - 2\dot{x} + (5/2)x = 0$ subject to $x(0) = 1$
3. $\ddot{x} + 2\dot{x} + 5x = 5$ subject to $x(0) = -1$

Substitute in $x(t) = Ae^{zt} + B$ to each of the equations and solve for each of the constants.

1. Substituting we have $Ae^{zt}(z+2) + 2B = 4$ as there is no e^{zt} on the RHS $z+2 = 0$ i.e. $z = -2$, $A = -1$ and $B = 2$.
2. Substituting
$$4Az^2e^{zt} - 2Aze^{zt} + (5/2)Ae^{zt} + (5/2)B = 0$$
i.e.
$$8Az^2e^{zt} - 4Aze^{zt} + 5Ae^{zt} + 5B = 0$$
By inspection constant terms on LHS must equal constant terms on RHS so either $B = 0$ if $z \neq 0$ or $5A + 5B = 0$ if $z = 0$ which contradicts the initial condition $A + B = 1$. Thus $z \neq 0$, $A = 1$ and $B = 0$. Substituting in these values z must be a root of $8z^2 - 4z + 5$ so $z = 1/4 + 3j/4$ or $1/4 - 3j/4$.
3. As above $A = -2, B = 1$ and $z = -1 + 2j$ or $-1 - 2j$.



Left-hand is Part 2 and right-hand is Part 3. Point out if unstable diverges from equilibrium which is at 'B'.

4: Application – RLC Circuits

Consider the RLC series circuit shown in Figure 1 where the input $u(t) = v(t)$ is the voltage of the supply and the output $y(t)$ is the voltage across the capacitor $y(t) = V_C(t)$ with $y(0) = 10\,\mathrm{V}$ and $i(0) = -5\,\mathrm{A}$.

1. Show the output satisfies

$$LC\ddot{y}(t) + RC\dot{y}(t) + y(t) = u(t) \tag{1}$$

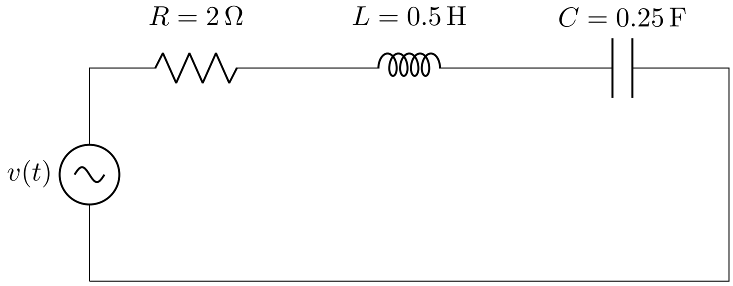


Figure 1: RLC series circuit

2. Let
$$z = \left(-R \pm \sqrt{R^2 - 4L/C}\right)/(2L)$$
and $a = Re(z)$ and $b = Im(z)$. Suppose $u(t) = 0$. Prove that $y(t) = A\exp((a \pm jb)t)$ are solutions to (1) for all $t \geq 0$.

3. Plot the trajectory of $y(t)$ in the complex plane. What would happen to this trajectory if $R = 0$ or $R < 0$.

4. Hence, prove that $y(t) = Be^{at}\cos(bt + \phi)$ is a solution to (1). **Hint:** if $y_1(t)$ and $y_2(t)$ are solutions to (1) then $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution.

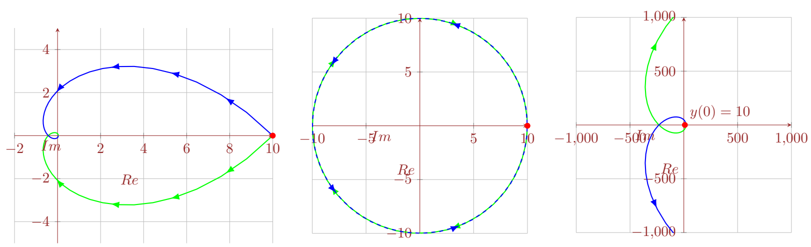
5. Find and plot the current $i(t)$ as a function of t.

1. Follows by KVL and is covered in lectures. Gives the equation $\ddot{y} + 4\dot{y} + 8y = 8u(t)$.

2. z is the roots of the characteristic of (1). Substituting Ae^{zt} into (1) we have
$$(LCz^2 + RCz + 1)Ae^{zt} = 0$$
which is true if and only if z is a root of the characteristic of (1) or $A = 0$. From initial condition $y(0) = 10 = A$ we know $A \neq 0$.

   Or more numerically, $z = -2 \pm 2j$. Check by substituting $A\exp((-2 \pm 2j)t)$ into $\ddot{y} + 4\dot{y} + 8y = 0$. $A = 10$ from initial condition.

3. Should see similar to Q3 but with stable (spiral into origin), marginally stable (circle/ellipse) and unstable (spiral out). Figures from left to right $R = 2, 0$ and -2.



4. Suppose $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution i.e.

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = 10e^{-2t}\left((c_1 + c_2)\cos(2t) + j(c_1 - c_2)\sin(2t)\right)$$

As $y(t)$ is a solution the initial condition says $y(0) = 10 = 10(c_1 + c_2)$ i.e. $c_1 + c_2 = 1$. Since $y_2 = \overline{y_1}$, requiring $y(t)$ to be real for all t forces $c_2 = \overline{c_1}$, and then $c_1 + c_2 = 2Re(c_1) = 1$ gives $Re(c_1) = Re(c_2) = 1/2$. To find imaginary part we use the initial condition $i(0) = -5$. As

$$\begin{aligned}\dot{y}(t) = \frac{i(t)}{C} &= 4i(t)\\ &= 20e^{-2t}\Big(-(c_1 + c_2)\big(\cos(2t) + \sin(2t)\big) + j(c_1 - c_2)\big(\cos(2t) - \sin(2t)\big)\Big)\end{aligned}$$

Using the initial condition $\dot{y}(0) = -20$

$$-20 = 20\left(-c_1 - c_2 + j(c_1 - c_2)\right)$$

As c_1 and c_2 are complex conjugates this implies $Im(c_1) = Im(c_2) = 0$. i.e. $c_1 = c_2 = 1/2$. NB a more typical way to solve this would be to solve the simultaneous equations from the initial conditions i.e. solve

$$\begin{cases} 1 = c_1 + c_2 \\ -1 = -c_1 - c_2 + j(c_1 - c_2) \end{cases}$$

simultaneously **or METHOD B**

$$\dot{y}(t) = c_1 \dot{y}_1(t) + c_2 \dot{y}_2(t) = z c_1 y_1(t) + \overline{z} c_2 y_2(t)$$

As $\dot{y}(t) = i(t)/C$ from the initial conditions we know $y_1(0) = y_2(0) = 10$ and $\dot{y}(0) = -20$. So

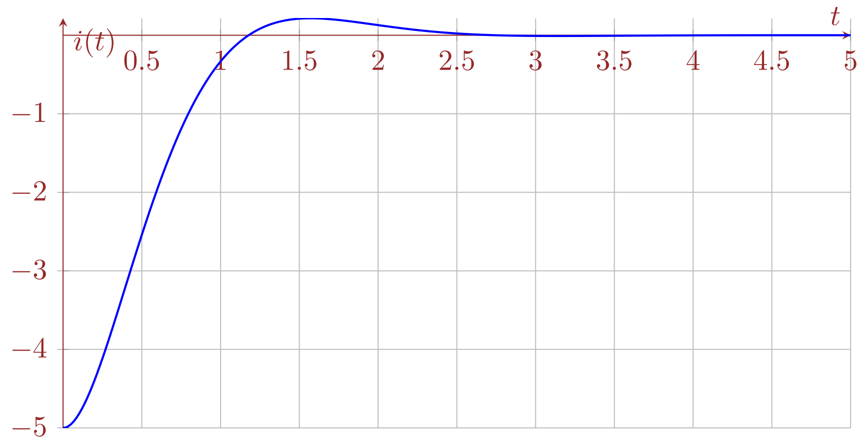
$$c_1 z + c_2 \overline{z} = -2 \quad \text{which, together with } c_1 + c_2 = 1, \text{ gives } c_1 = c_2 = \tfrac{1}{2}.$$

Thus

$$y(t) = 5(e^{(-2-2j)t} + e^{(-2+2j)t}) = 10e^{-2t}\cos(2t)$$

is a solution.

5. Current $i(t) = C\dot{y}(t) = -5e^{-2t}(\cos(2t) + \sin(2t)) = -5\sqrt{2}e^{-2t}\cos(2t - \pi/4)$



5: Energy and Power

1. The periodic signal $f(t)$ in Figure 2 is an idealised version of the pulse train emitted by the Kurnell weather radar. Calculate its energy, power and RMS.

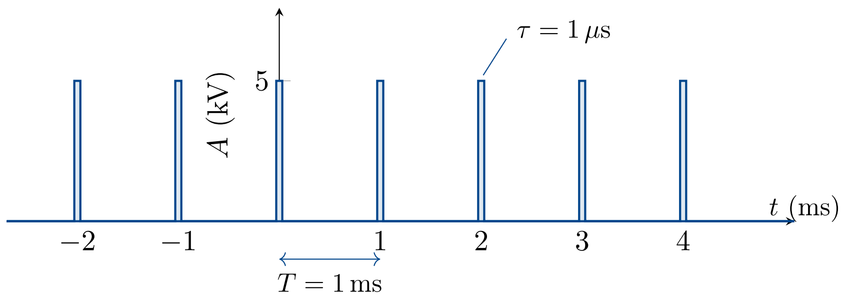
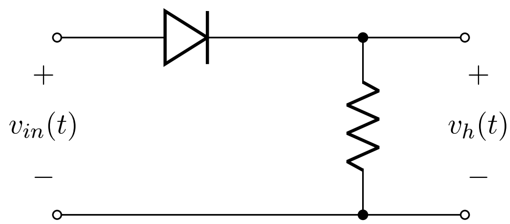


Figure 2: Pulse-train emitted by the Kurnell weather radar, $f(t)$ – excluding the carrier frequency to be covered in Week 11.

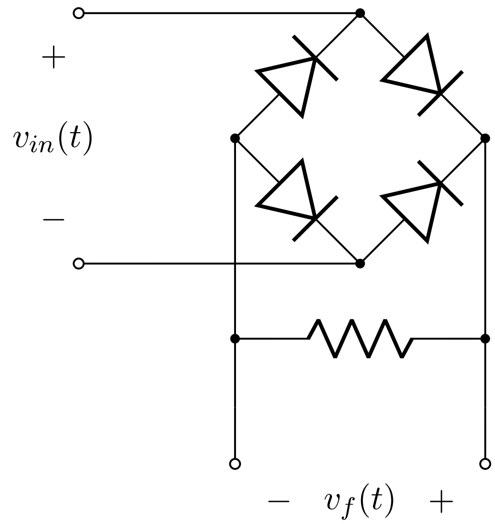
2. Find the energy and power of the functions
   (a) $f(t) + 1$
   (b) $f(t + \tau)$ for $\tau > 0$
   (c) **[Extension]** $e^{-|t|} f(t)$
3. What happens to the energy of a periodic signal if the power is non-zero?

4. Find the energy, power and RMS of and sketch the output, $y(t)$:
   (a) of the half-wave rectifier circuit in Figure 3a, i.e. $y(t) = v_h(t)$
   (b) of the full-wave rectifier in Figure 3b, i.e. $y(t) = v_f(t)$.

   Assume the input $u(t) = v_{in}(t) = \sin(100\pi t)$ for both circuits and all diodes are ideal.



(a) Half-wave rectifier



(b) full-wave rectifier

⋆ 5. Find the energy, power and RMS for $v_h(t)$ and $v_f(t)$ above but assuming that the diodes have a threshold voltage of 0.7 V.

From lectures

$E(f) = \langle f, f\rangle = \int_{-\infty}^{\infty} |f(t)|^2\, dt$ and $P(f) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|^2\, dt$ and $RMS(f) = \sqrt{P(f)}$

If $f(t)$ has period T i.e. $f(t+T) = f(t)$. Then

$$P(f) = \frac{1}{T}\int_0^T |f(t)|^2\, dt = \langle f, f\rangle_P \text{ and } RMS(f) = \sqrt{\langle f, f\rangle_P} = \|f\|_P$$

1. $f(t)$ is periodic with period $T = 1$ ms and in each period $f(t) = 5$ kV for a duration of $\tau = 1\,\mu$s $= 10^{-3}$ ms.

Energy: will be unbounded but let's do the calculation anyway

$$
\begin{aligned}
E(f) = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \lim_{k\to\infty} \sum_{n=-k}^{k} \int_{nT-T/2}^{(n+1)T-T/2} 5^2 p_\tau(t-nT)\, dt \\
= \lim_{k\to\infty} \sum_{n=-k}^{k} 25\tau \\
= 25 \left(\lim_{k\to\infty} (2k+1)\tau \right)
\end{aligned}
$$

which is unbounded.

Power:

$$
\begin{aligned}
P(f) = \frac{1}{T}\int_0^T |f(t)|^2 \, dt &= \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2 \, dt \\
&= 5^2 \frac{\tau}{T} \\
&= 25 \times 10^{-3} \, (\mathrm{kV})^2 \\
&= 2.5 \times 10^4 \, \mathrm{V}^2
\end{aligned}
$$

RMS: which implies $RMS(f) = \sqrt{P(f)} = 5 \times \sqrt{10}^3 \approx 158\,\mathrm{V}$.

2. (a) For $f(t) + 1$

Energy: remains unbounded

$$
\begin{aligned}
E(f+1) = \int_{-\infty}^{\infty} |f(t)+1|^2 \, dt &= \int_{-\infty}^{\infty} (f(t))^2 + 2f(t) + 1 \, dt \\
&= \int_{-\infty}^{\infty} (f(t))^2 \, dt + \int_{-\infty}^{\infty} 2f(t)\, dt + \int_{-\infty}^{\infty} 1 \, dt
\end{aligned}
$$

all of which are positive and divergent. NB technically this means the final equality is not correct. Actual solution involves replacing limits with finite b solving each integral, simplifying and then taking the limit $b \to \infty$.

Power: Similarly to the above expansion:

$$
\begin{aligned}
P(f+1) &= \frac{1}{T}\int_{-T/2}^{T/2} (f(t))^2 \, dt + \frac{1}{T}\int_{-T/2}^{T/2} 2f(t)\, dt + \frac{1}{T}\int_{-T/2}^{T/2} 1 \, dt \\
&= P(f) + \frac{10 \times 10^3 \tau}{T} + 1 \\
&= P(f) + 11 \\
&\approx 2.5 \times 10^4 \, \mathrm{V}^2
\end{aligned}
$$

RMS: $RMS(f(t)+1) = \sqrt{P(f+1)} \approx 158\,\mathrm{V}$.

(b) Energy, power and RMS for a periodic signal are invariant under time-shift.

(c) Note $e^{-|t|}f(t)$ is not periodic.

Energy: We can represent

$$f(t) = 5 \sum_{n=-\infty}^{\infty} p_\tau(t - nT)$$

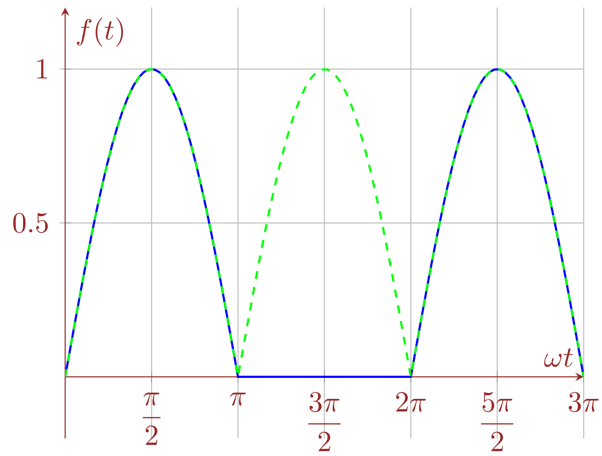
The total energy will be the sum of the energy of each of the pulses. Write $A = 5\,\mathrm{kV} = 5 \times 10^3\,\mathrm{V}$ for the pulse amplitude.

$$\begin{aligned}
E(e^{-|t|}f(t)) &= \int_{-\infty}^{\infty} e^{-2|t|} f(t)^2 \, dt \\
&= A^2 \sum_{n=-\infty}^{\infty} \int_{nT-\tau/2}^{nT+\tau/2} e^{-2|t|} \, dt \\
&= A^2 \left(2 \int_0^{\tau/2} e^{-2t} \, dt + 2 \sum_{n=1}^{\infty} \int_{nT-\tau/2}^{nT+\tau/2} e^{-2t} \, dt \right) \\
&= A^2 \left(\left(1 - e^{-\tau}\right) + \left(e^{\tau} - e^{-\tau}\right) \sum_{n=1}^{\infty} e^{-2nT} \right) \\
&\approx \left(25 \times 10^6\right) \left(1 \times 10^{-6} + 2 \times 10^{-6} \times 500\right) \\
&\approx \left(25 \times 10^6\right) \left(1.001 \times 10^{-3}\right) \\
&\approx 2.5 \times 10^4 \,\mathrm{V^2 s}
\end{aligned}$$

Power: will be 0 as the total energy is finite i.e. because the energy per pulse heads to 0.

3. Finite energy implies zero power, so non-zero power implies unbounded energy.

4. Assuming all diodes are ideal means no voltage drop across the diode. So $v_h(t) = \max\{0, \sin(100\pi t)\}$ will be a half-rectified sin wave and $v_f(t) = |\sin(\omega t)|$ will be a fully rectified sin wave, with $\omega = 100\pi$.



We see $2E(v_h) = E(v_f)$ and $2P(v_h) = P(v_f)$ as

$$\int_0^{2\pi/\omega} |v_f(t)|^2 \, dt = 2\int_0^{\pi/\omega} |v_h(t)|^2 \, dt = 2\int_0^{2\pi/\omega} |v_h(t)|^2 \, dt$$

From $E(v_f)$ and $P(v_f)$ we find $E(v_h)$ is unbounded and $P(v_h) = 1/4$. So it is not necessary to find energy and power for both $v_h(t)$ and $v_f(t)$.

(a) Energy and power for $v_h(t)$.

Energy: Unbounded.

Power: Period of $v_h(t)$ is $T = 2\pi/\omega$.

$$\begin{aligned}
P(v_h) &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} |v_h(t)|^2 \, dt \\
&= \frac{\omega}{2\pi}\left(\int_0^{\pi/\omega} |v_h(t)|^2 \, dt + \int_{\pi/\omega}^{2\pi/\omega} |v_h(t)|^2 \, dt\right) \\
&= \frac{\omega}{2\pi}\int_0^{\pi/\omega} |\sin(\omega t)|^2 \, dt \\
&= \frac{\omega}{4\pi}\int_0^{\pi/\omega} 1 - \cos(2\omega t) \, dt \\
&= \frac{\omega}{4\pi}\left(t - \frac{\sin(2\omega t)}{2\omega}\right|_0^{\pi/\omega} \\
&= \frac{\omega}{4\pi}\left(\frac{\pi}{\omega}\right) \\
&= 1/4
\end{aligned}$$

RMS: $RMS(v_h) = \sqrt{1/4} = 1/2\,\mathrm{V}$.

(b) Energy and power for $v_f(t)$.

Energy: Unbounded.

Power: The fundamental period of $v_f(t)$ is $T = \pi/\omega$ because of the absolute value.

$$\begin{aligned}
P(v_f) &= \frac{\omega}{\pi}\int_0^{\pi/\omega} |\sin(\omega t)|^2\, dt \\
&= \frac{\omega}{\pi}\int_0^{\pi/\omega} \sin^2(\omega t)\, dt \\
&= \frac{\omega}{2\pi}\int_0^{\pi/\omega} 1 - \cos(2\omega t)\, dt \\
&= \frac{\omega}{2\pi}\left(t - \frac{\sin(2\omega t)}{2\omega}\right|_0^{\pi/\omega} \\
&= \frac{\omega}{2\pi}\left(\frac{\pi}{\omega}\right) \\
&= 1/2
\end{aligned}$$

RMS: $RMS(v_f) = 1/\sqrt{2} \approx 0.707\,\mathrm{V}$. Just like for $\cos(\omega t)$ found in lectures :)

5. Non-ideal diodes mean $v_h(t) = \max\{0, \sin(\omega t) - 0.7\}$ and $v_f(t) = \max\{0, |\sin(\omega t)| - 1.4\} = 0$. So energy, power and RMS of $v_f(t)$ are 0. For $v_h(t)$ the period $T = 2\pi/\omega$. Need to find the intervals $[t_D, t'_D]$ on which $v_h(t) \neq 0$ i.e. to solve $\sin(\omega t_D) = V_D = 0.7$.

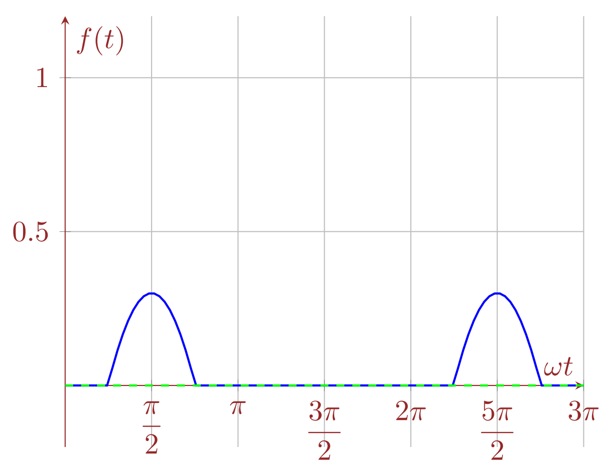
$$t_D = \arcsin(V_D)/\omega = \arcsin(0.7)/(100\pi) = 0.247 \times 10^{-2}$$

So $v_h(t) \neq 0$ for all $t \in [t_D, \pi/\omega - t_D] = [0.247, 0.753] \times 10^{-2}$. Let $t'_D = \pi/\omega - t_D$.

Similarly to Part 4a

$$
\begin{aligned}
P(v_h) &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} |v_h(t)|^2\,dt \\
&= \frac{\omega}{2\pi}\left(\int_0^{t_D} |v_h(t)|^2\,dt + \int_{t_D}^{t'_D} |v_h(t)|^2\,dt + \int_{t'_D}^{2\pi/\omega} |v_h(t)|^2\,dt\right) \\
&= \frac{\omega}{2\pi}\int_{t_D}^{t'_D} |\sin(\omega t) - V_D|^2\,dt \\
&= \frac{\omega}{2\pi}\int_{t_D}^{t'_D} (\sin(\omega t) - V_D)^2\,dt \\
&= \frac{\omega}{2\pi}\int_{t_D}^{t'_D} \sin^2(\omega t) - 2V_D\sin(\omega t) + V_D^2\,dt \\
&= \frac{\omega}{2\pi}\left(\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega} + \frac{2V_D\cos(\omega t)}{\omega} + V_D^2 t\right|_{t_D}^{t'_D} \\
&= \frac{\omega}{2\pi}\left(t\left(\frac{1}{2} + V_D^2\right) - \frac{\sin(2\omega t)}{4\omega} + \frac{2V_D\cos(\omega t)}{\omega}\right|_{t_D}^{t'_D} \\
&= 50\left(0.99\,t - \frac{\sin(200\pi t)}{400\pi} + \frac{1.4\cos(100\pi t)}{100\pi}\right|_{0.247\times10^{-2}}^{0.753\times10^{-2}} \\
&= 50\left(5.070\times10^{-3} - 4.830\times10^{-3}\right) \\
&\approx 1.2\times10^{-2}\,\mathrm{V}^2
\end{aligned}
$$

so $RMS(v_h) = \sqrt{1.2\times10^{-2}} \approx 0.109\,\mathrm{V}$, and the energy is unbounded.



6: Dirac δ-function

1. For any $\varepsilon > 0$ find the energy and power of the signal

$$f(\varepsilon, t) = \begin{cases} 1/\varepsilon, & |t| \le \varepsilon/2 \\ 0, & \text{otherwise} \end{cases}$$

what happens to the energy and power as $\varepsilon \to 0$?

$\star$ 2. Use $f(\varepsilon, t)$ to prove that $\delta(\alpha t) = \delta(t)/|\alpha|$.

1. We have

$$|f(t)|^2 = \begin{cases} 1/\varepsilon^2, & |t| \leq \varepsilon/2, \\ 0, & \text{otherwise} \end{cases}$$

Energy

$$E(f) = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{\varepsilon^2} \int_{-\varepsilon/2}^{\varepsilon/2} dt = 1/|\varepsilon|$$

Power

$$P(f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 \, dt = \frac{1}{|\varepsilon|} \lim_{T \to \infty} \frac{1}{2T} = 0$$

which also implies $RMS = 0$. As $\varepsilon \to 0^+$ energy is unbounded and power remains at 0. Not well defined as the δ function is not in L^2.

2. Main objective is to prove the scaling property of the δ function

$$f(\varepsilon, \alpha t) = \begin{cases} \frac{1}{\varepsilon}, & |\alpha t| \leq \frac{\varepsilon}{2} \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{\varepsilon}, & |t| \leq \frac{\varepsilon}{2|\alpha|} \\ 0, & \text{otherwise} \end{cases} = \frac{1}{|\alpha|} \begin{cases} \frac{|\alpha|}{\varepsilon}, & |t| \leq \frac{\varepsilon}{2|\alpha|} \\ 0, & \text{otherwise} \end{cases} = \frac{1}{|\alpha|} f\left(\frac{\varepsilon}{|\alpha|}, t\right)$$

Setting $\varepsilon^* = \varepsilon/|\alpha|$ and taking the limit:

$$\delta(\alpha t) = \lim_{\varepsilon \to 0^+} f(\varepsilon, \alpha t) = \frac{1}{|\alpha|} \lim_{\varepsilon^* \to 0^+} f(\varepsilon^*, t) = \frac{1}{|\alpha|} \delta(t)$$