Week 1 — Lecture notes

The core: exponentials, Euler's formula and the complex plane

The figures below are snapshots of interactive widgets. The live versions, where you can move the sliders yourself, are at tchaffey.com/elec2302.

Signals & Systems lives at the intersection of **differential equations**, **linear algebra**, **complex numbers** and **trigonometry**.

(a) The exponential function

For $z \in \mathbb{C}$, define the exponential by its power series:

$$e^z := 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Properties. First, $e^0 = 1$ (straight from the definition) (1)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z} e^z &= 0 + 1 + \frac{2z}{2!} + \frac{3z^2}{3!} + \cdots \\ &= 1 + z + \frac{z^2}{2!} + \cdots \\ &= e^z\end{aligned} \tag{2}$$

Combining the two properties:

$$\boxed{f(z) = f'(z), \qquad f(0) = 1}$$

It turns out $f(z) = e^z$ is the unique solution to this differential equation, and in fact the differential equation can be used as an alternative definition.

Extension of (2). By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{zt} = 0 + z + \frac{2t\,z^2}{2!} + \frac{3t^2 z^3}{3!} + \cdots = z\,e^{zt}.$$

(b) Euler's formula

Let's now represent sinusoids as complex numbers. Let

$$g(\theta) = \frac{\cos\theta + j\sin\theta}{e^{j\theta}}.$$

Then

$$\begin{aligned}g'(\theta) &= \big(-\sin\theta + j\cos\theta\big)e^{-j\theta} - \big(\cos\theta + j\sin\theta\big)\,j\,e^{-j\theta} \\ &= \big(-\sin\theta + j\cos\theta\big)e^{-j\theta} - \big(-\sin\theta + j\cos\theta\big)e^{-j\theta} \\ &= 0.\end{aligned}$$

So $g(\theta) = C$ (a constant):

$$C\, e^{j\theta} = \cos\theta + j\sin\theta.$$

Set $\theta = 0$. Then $C = 1$, so we have

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

What does this mean graphically? $e^{j\theta}$ is the unit vector at angle θ; its real part is $\cos\theta$ and its imaginary part is $\sin\theta$. Advance θ and the vector rotates, its shadow tracing a sinusoid.

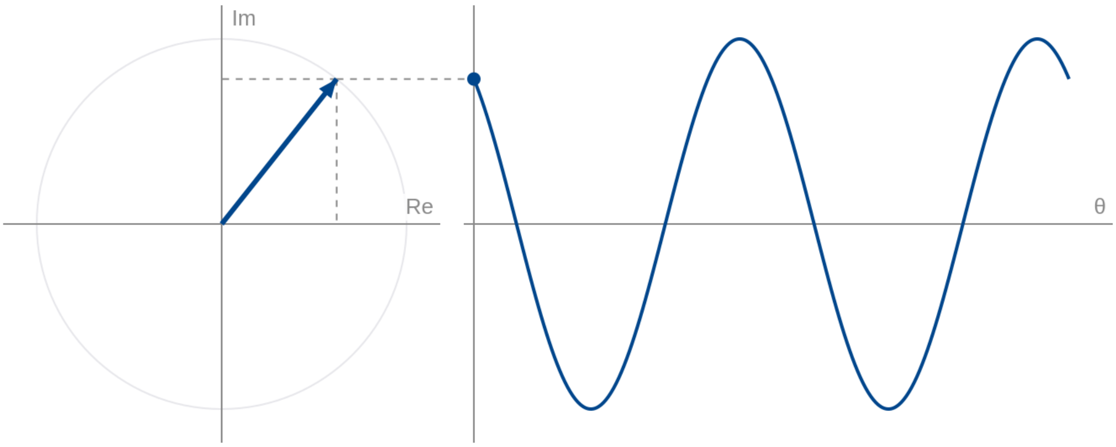

Figure 1: The unit vector $e^{j\theta}$ and its imaginary part unrolling into a sinusoid.

In particular: $e^{j\pi/2} = j, \qquad e^{j\pi} = -1.$

Manipulating sinusoids using complex numbers

Why use complex numbers to represent sinusoids? Because it makes the algebra easier. For example,

$$x_1(t) = 3\cos\omega t, \qquad x_2(t) = 4\cos(\omega t + \tfrac{\pi}{2}), \qquad x_1(t) + x_2(t) = \ ?$$

The direct way uses the identity $\cos(\theta + \frac{\pi}{2}) = -\sin\theta$, so $x_2(t) = -4\sin\omega t$, giving

$$3\cos\omega t - 4\sin\omega t \ \equiv\ A\cos(\omega t + \varphi) \quad \longrightarrow \quad \text{match coefficients}\ldots$$

With complex numbers:

$$\begin{aligned}
x_1(t) &= \mathrm{Re}(3\, e^{j\omega t}) \\
x_2(t) &= \mathrm{Re}(4\, e^{j\omega t + j\pi/2}) = \mathrm{Re}(4\, e^{j\pi/2}\, e^{j\omega t}) = \mathrm{Re}(4j\, e^{j\omega t}) \\
x_1(t) + x_2(t) &= \mathrm{Re}(3\, e^{j\omega t} + 4j\, e^{j\omega t}) \\
&= \mathrm{Re}((3 + 4j)\, e^{j\omega t}) \\
&= \mathrm{Re}(5\, e^{j0.927}\, e^{j\omega t}) = 5\cos(\omega t + 0.927).
\end{aligned}$$

The triangle $3 + 4j = 5\angle 53.1^\circ$ rotates.

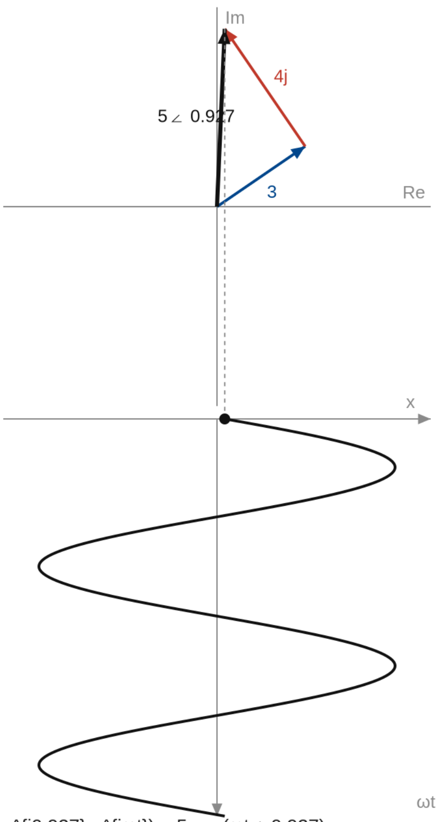

Figure 2: Adding 3 and $4j$ tip to tail. The resultant $5\angle 0.927$ rotates, and its real part traces $5\cos(\omega t + 0.927)$.

(c) Euler's formula & differential equations

Another differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,x(t) = j\,x(t) \qquad (3)$$

We've already seen $\dfrac{\mathrm{d}}{\mathrm{d}t}e^{zt} = z\,e^{zt}$. Set $z = j$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,e^{jt} = j\,e^{jt}.$$

This solves (3). What does it look like in $\mathbb{C}$, and what does the real part look like? What about other choices of z?

$$z = -1: \quad \frac{\mathrm{d}}{\mathrm{d}t}\,e^{-t} = -e^{-t}. \qquad \text{No imaginary component.}$$

$$z = -1 + j: \quad \frac{\mathrm{d}}{\mathrm{d}t}\,e^{(-1+j)t} = (-1+j)\,e^{(-1+j)t}.$$

The widget below plots the trajectory of e^{zt} for several values of z.

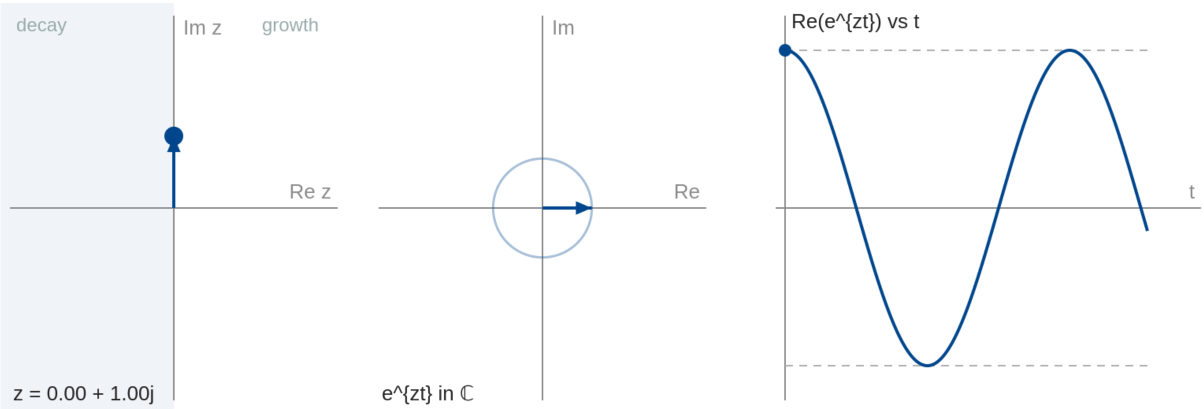

Figure 3: Left: the position of z. Middle: the trajectory of e^{zt} in $\mathbb{C}$. Right: its real part against t. Shown for $z = j$, giving pure oscillation.

The position of z determines the behaviour: oscillations, decay, growth.

(d) What has this got to do with electrical engineering?

Example: an RLC circuit (the **system**) with a sinusoidal voltage input (the **signal**).

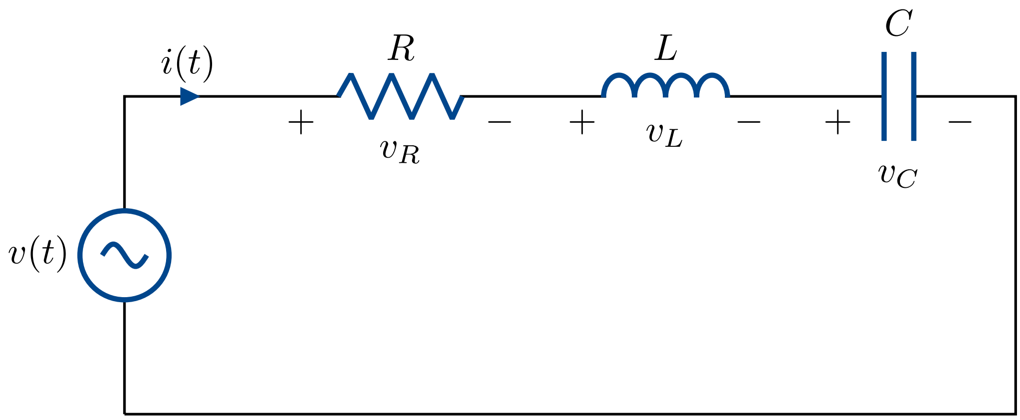

Figure 4: Series RLC circuit.

KVL around the loop:

$$v(t) = v_R + v_L + v_C$$

$$v(t) = R\,i(t) + L\,\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,\mathrm{d}\tau \tag{4}$$

with the lower limit taken to $-\infty$ for the steady-state response. Let $v(t) = V\cos\omega t = \mathrm{Re}(Ve^{j\omega t})$ and look for a current of the same form, $i(t) = \mathrm{Re}(Ie^{j\omega t})$.

Substitute in (4):

$$\begin{aligned} Ve^{j\omega t} &= RIe^{j\omega t} + L\frac{\mathrm{d}}{\mathrm{d}t}Ie^{j\omega t} + \frac{1}{C}\int Ie^{j\omega\tau}\,\mathrm{d}\tau \\ &= RIe^{j\omega t} + j\omega L\,Ie^{j\omega t} + \frac{I}{j\omega C}\,e^{j\omega t} + \frac{K}{C}, \end{aligned}$$

where K is a constant of integration. Matching coefficients of $e^{j\omega t}$, we find $K = 0$ (K represents a residual charge on the capacitor, which a sinusoidal voltage is not able to maintain at steady state).

Divide through by $e^{j\omega t}$ (never zero):

$$V = I\underbrace{\left(R + j\omega L + \tfrac{1}{j\omega C}\right)}_{\text{impedance } Z(j\omega)} \qquad \Longrightarrow \qquad I = \frac{V}{Z}.$$

Suppose $R = 30\,\Omega$ and $\omega L - \frac{1}{\omega C} = 40\,\Omega$. Then

$$Z = 30 + 40j = 50\,e^{j0.927}, \qquad I = \frac{V}{50\,e^{j0.927}},$$

$$i(t) = \mathrm{Re}(Ie^{j\omega t}) = \frac{V}{50}\cos(\omega t - 0.927).$$

The widget below plots input and output sinusoids for several values of the reactance Z. For some values, the output lags the input, while for others, the output leads.

The RLC circuit maps a sinusoid to a sinusoid with the **same frequency**. This is the foundation of the rest of the course.

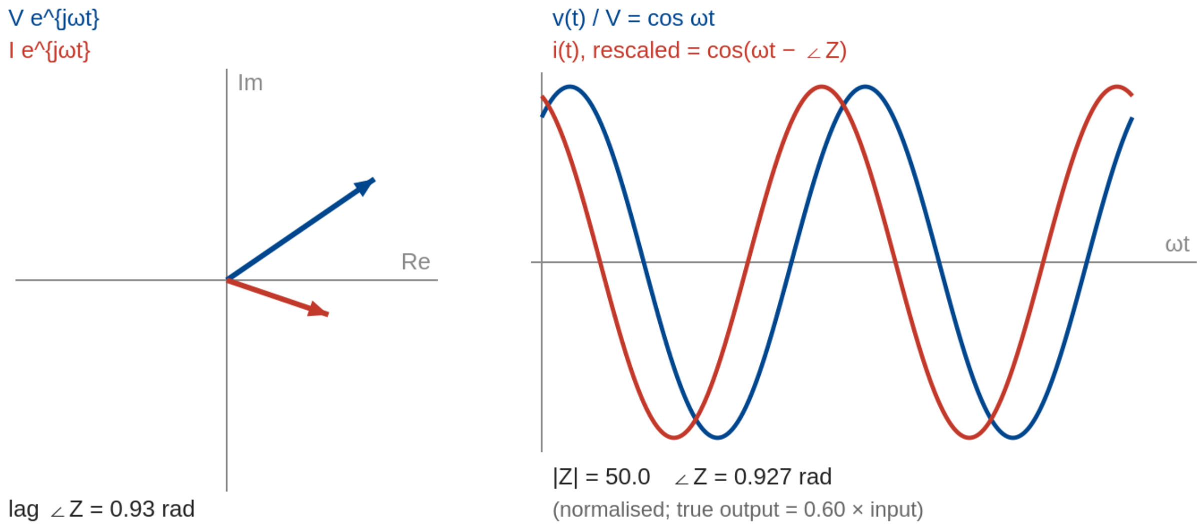

Figure 5: Input and output phasors (left) and waveforms (right), at $R = 30\,\Omega$ and $\omega L - 1/\omega C = 40\,\Omega$, so $|Z| = 50\,\Omega$ and $\angle Z = 0.927$. Amplitudes are normalised here; the live version can show the output at its true size, $R/|Z| = 0.60$ of the input.