

ELEC2302/9302 Signals and Systems – Quiz 3 practice questions, answers and feedback¹

How to use this question bank: *don't* work through the questions from beginning to end. Start by doing one question from each group, identify the areas where you are less confident, and work on those first.

The correct option is marked ✓.

1. The Laplace transform (week 8)

Laplace transform: definition, ROC and table pairs

1. What is the region of convergence of the Laplace transform of $u(t) = (3e^{-4t} - 2e^t) H(t)$?

(a) $\text{Re}(s) > 1$

✓

(b) $-4 < \text{Re}(s) < 1$

(c) $\text{Re}(s) > 0$

(d) $\text{Re}(s) > -4$

Feedback: The transform of a sum converges only where the transform of every term converges, so the ROC is the intersection of the individual ROCs.

(a) Correct. $U(s) = \frac{3}{s+4} - \frac{2}{s-1}$, and the intersection of $\text{Re}(s) > -4$ and $\text{Re}(s) > 1$ is $\text{Re}(s) > 1$.

(b) A strip is the ROC of a two-sided signal, but this signal is zero for $t < 0$, so its ROC extends to the right.

(c) $\text{Re}(s) > 0$ is the ROC of $H(t)$ alone, but the term $e^t H(t)$ needs $\text{Re}(s) > 1$.

(d) This is the ROC of the decaying term $3e^{-4t} H(t)$ alone, and the growing term $e^t H(t)$ does not converge there.

2. The signal $u(t) = e^{(-2+5j)t} H(t)$ has Laplace transform $U(s) = \frac{1}{s+2-5j}$. What is its region of convergence?

(a) $\text{Re}(s) > 5$

(b) $\text{Re}(s) > -2$

✓

(c) $|s| > \sqrt{29}$

(d) $\text{Re}(s) > 2$

Feedback: The ROC of $e^{s_0 t} H(t)$ is $\text{Re}(s) > \text{Re}(s_0)$, because only the real part of the exponent affects the size of $u(t)e^{-st}$.

(a) This uses the imaginary part of the exponent, which only rotates the signal and does not change its size.

(b) Correct. $|u(t)e^{-\sigma t}| = e^{(-2-\sigma)t}$, which decays when $\sigma > -2$.

¹Some questions are adapted from B. P. Lathi and R. A. Green, *Linear Systems and Signals*, 3rd ed. (chapters 4 and 7), from ELEC3004 *Signals, Systems and Control* at the University of Queensland (2017), and from S. Boyd, EE102 *Introduction to Signals and Systems*, Stanford University.

- (c) The ROC is bounded by a vertical line $\text{Re}(s) = \text{constant}$, not by a circle, and $\sqrt{29}$ is the modulus of the pole.
- (d) The pole is at $s = -2 + 5j$, so the boundary is $\text{Re}(s) = -2$, not $+2$.
3. The centered pulse $p_2(t)$ is delayed by 3 s, giving $u(t) = p_2(t - 3)$. What is $U(s)$?
- (a) $\frac{e^{-s} - e^{-5s}}{s}$
- (b) $\frac{e^{-3s}(1 - e^{-2s})}{s}$
- (c) $\frac{e^{-4s} - e^{-2s}}{s}$
- (d) $\frac{e^{-2s} - e^{-4s}}{s}$ ✓

Feedback: Find where the delayed pulse is on, write it as a difference of two delayed steps, and use $H(t - \tau) \leftrightarrow e^{-s\tau}/s$.

- (a) This is a pulse on $1 < t < 5$, which has width 4 instead of width 2.
- (b) This is a pulse on $3 < t < 5$, which starts at $t = 3$ instead of being centered there.
- (c) This is $H(t - 4) - H(t - 2)$, which is the negative of the pulse.
- (d) Correct. $p_2(t - 3)$ is 1 on $2 < t < 4$, so $u(t) = H(t - 2) - H(t - 4)$.
4. What is the Laplace transform of $u(t) = \cos(3t) \cos(t)H(t)$?

- (a) $\frac{1}{2} \left(\frac{s}{s^2 + 9} + \frac{s}{s^2 + 1} \right)$
- (b) $\frac{s^2}{(s^2 + 9)(s^2 + 1)}$
- (c) $\frac{1}{2} \left(\frac{s}{s^2 + 16} + \frac{s}{s^2 + 4} \right)$ ✓
- (d) $\frac{1}{2} \left(\frac{s}{s^2 + 16} - \frac{s}{s^2 + 4} \right)$

Feedback: Rewrite the product as a sum with $\cos A \cos B = \frac{1}{2} (\cos(A + B) + \cos(A - B))$, then transform each term.

- (a) This transforms $\frac{1}{2}(\cos 3t + \cos t)$, a sum of the original cosines rather than their product.
- (b) This multiplies the two transforms, but a product in time is not a product of transforms.
- (c) Correct. $\cos 3t \cos t = \frac{1}{2}(\cos 4t + \cos 2t)$, and each cosine uses the pair $s/(s^2 + \omega^2)$.
- (d) The product-to-sum identity for two cosines has a plus sign, not a minus sign.
5. A system has transfer function $G(s) = 5$ (a constant). What is its impulse response $h(t)$?
- (a) $5H(t)$
- (b) It has no impulse response because it is not a dynamic system.
- (c) 5 for all t

(d) $5\delta(t)$

✓

Feedback: The impulse response is the inverse transform of $G(s)$, and $\mathcal{L}\{\delta(t)\} = 1$.

(a) $5H(t)$ has transform $5/s$, not 5.

(b) Every LTI system has an impulse response, and for a static gain it is a scaled impulse.

(c) A constant signal for all t is not the inverse of a constant transform, which is an impulse.(d) Correct. $h(t) = 5\delta(t)$, which is the pure gain $y(t) = 5u(t)$.

6. Write $\cos(4t)H(t) = \frac{1}{2}e^{4jt}H(t) + \frac{1}{2}e^{-4jt}H(t)$ and use linearity with $\mathcal{L}\{e^{s_0t}H(t)\} = 1/(s - s_0)$. The result is

(a) $\frac{4}{s^2 + 16}$ (b) $\frac{1}{s^2 + 16}$ (c) $\frac{s}{s^2 + 16}$

✓

(d) $\frac{s}{s^2 - 16}$

Feedback: Transform each exponential with $s_0 = \pm 4j$ and put the two fractions over the common denominator $(s - 4j)(s + 4j) = s^2 + 16$.

(a) This is the transform of $\sin(4t)H(t)$, not $\cos(4t)H(t)$.(b) The numerator is $\frac{1}{2}((s + 4j) + (s - 4j)) = s$, not 1.(c) Correct. $\frac{1}{2}\left(\frac{1}{s-4j} + \frac{1}{s+4j}\right) = \frac{1}{2} \cdot \frac{2s}{s^2+16} = \frac{s}{s^2+16}$.(d) The denominator is $(s - 4j)(s + 4j) = s^2 - (4j)^2 = s^2 + 16$, because $(4j)^2 = -16$.

Laplace transform properties

7. Which signal has Laplace transform $\frac{e^{-3s}}{s+1}$?

(a) $e^{-(t+3)}H(t)$ (b) $e^{-(t-3)}H(t-3)$

✓

(c) $e^{-t}H(t-3)$ (d) $e^{-(t-3)}H(t)$

Feedback: The factor e^{-3s} means the whole signal $e^{-t}H(t)$ is delayed by 3, so both the exponential and the step are shifted.

(a) This is $e^{-3}e^{-t}H(t)$, whose transform has the constant e^{-3} rather than the delay factor e^{-3s} .(b) Correct. This is $e^{-t}H(t)$ delayed by 3, and its transform is $e^{-3s}/(s+1)$.(c) Only the step is shifted, so this is $e^{-3}e^{-(t-3)}H(t-3)$, which has an extra factor e^{-3} .(d) Only the exponential is shifted, so this is $e^3e^{-t}H(t)$ with transform $e^3/(s+1)$.

8. An LTI system, initially at rest, has step response $(e^{-t} - e^{-2t})H(t)$. What is its impulse response $h(t)$?

- (a) $\left(\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right)H(t)$
- (b) $(2e^{-2t} - e^{-t})H(t)$
- (c) $(e^{-t} - e^{-2t})H(t)$
- (d) $(e^{-t} - 2e^{-2t})H(t)$

✓

Feedback: The step response has transform $G(s)/s$, so the impulse response is the derivative of the step response.

- (a) This is the integral of the step response, not its derivative.
- (b) Correct. Differentiating gives $-e^{-t} + 2e^{-2t}$ for $t > 0$, and the step response is 0 at $t = 0$, so no impulse appears.
- (c) This is the step response itself, not the impulse response.
- (d) This is the negative of the derivative, since $\frac{d}{dt}e^{-t} = -e^{-t}$ and $\frac{d}{dt}(-e^{-2t}) = 2e^{-2t}$.

9. Let $y(t) = (e^{-t}H(t)) * (e^{-3t}H(t))$. Use the Laplace transform to find $y(1)$.

Answer: 0.1590

Feedback: Convolution in time is multiplication of transforms, so $Y(s) = \frac{1}{(s+1)(s+3)} = \frac{1/2}{s+1} - \frac{1/2}{s+3}$. Then $y(t) = \frac{1}{2}(e^{-t} - e^{-3t})H(t)$ and $y(1) = \frac{1}{2}(e^{-1} - e^{-3}) = 0.1590$.

10. What is the Laplace transform of $u(t) = e^{2t} \sin(3t)H(t)$?

- (a) $\frac{3}{s^2 - 4s + 7}$
- (b) $\frac{3}{s^2 - 4s + 13}$
- (c) $\frac{3}{s^2 + 4s + 13}$
- (d) $\frac{s - 2}{s^2 - 4s + 13}$

✓

Feedback: Start from $\sin(3t)H(t) \leftrightarrow 3/(s^2 + 9)$ and use the frequency shift $e^{at}f(t) \leftrightarrow F(s - a)$ with $a = 2$.

- (a) The constant term is $4 + \omega^2 = 4 + 9 = 13$, but this adds $\omega = 3$ instead of ω^2 .
- (b) Correct. $\frac{3}{(s-2)^2+9} = \frac{3}{s^2-4s+13}$.
- (c) This shifts the wrong way, giving the transform of $e^{-2t} \sin(3t)H(t)$.
- (d) This is the shifted cosine pair, the transform of $e^{2t} \cos(3t)H(t)$.

11. What is the Laplace transform of $y(t) = \int_0^t \cos(3\tau) d\tau$ for $t \geq 0$?

- (a) $\frac{1}{s^2 + 9}$

✓

- (b) $\frac{3}{s^2 + 9}$
- (c) $\frac{1}{s(s^2 + 9)}$
- (d) $\frac{s^2}{s^2 + 9}$

Feedback: Integrating from 0 to t divides the transform by s .

- (a) Correct. $\frac{1}{s} \cdot \frac{s}{s^2+9} = \frac{1}{s^2+9}$, which matches the transform of $y = \frac{1}{3} \sin 3t$.
- (b) This is the transform of $\sin 3t$, but the integral of $\cos 3\tau$ is $\frac{1}{3} \sin 3t$.
- (c) This divides by s twice, which is the transform of the integral of $y(t)$.
- (d) This multiplies by s , which is the rule for a derivative, not an integral.

12. A signal $q(t)$, zero for $t < 0$, has $Q(s) = \frac{2 - 4e^{-2s}}{s + 3}$. Find $q(3)$.

Answer: -0.1989

Feedback: The factor e^{-2s} delays the second term by 2, so $q(t) = 2e^{-3t}H(t) - 4e^{-3(t-2)}H(t-2)$. At $t = 3$ both terms are on, giving $q(3) = 2e^{-9} - 4e^{-3} = -0.1989$.

13. A system has transfer function $G(s) = k \left(1 + \frac{1}{\tau_i s} \right)$ with $k = 2$ and $\tau_i = 0.5$ s. Its input is $u(t) = H(t)$. Using the integral property, find the output $y(t)$ at $t = 3$ s.

Answer: 14.00

Feedback: With $k = 2$ and $\tau_i = 0.5$, $G(s) = 2 + 4/s$, so $Y(s) = 2/s + 4/s^2$. Each $1/s$ integrates once, so $y(t) = (2 + 4t)H(t)$ and $y(3) = 14.00$.

Solving differential equations with initial conditions

14. $y(t)$ satisfies $\ddot{y} + 16y = 0$ for $t \geq 0$, with $y(0) = 0$ and $\dot{y}(0) = 2$. What is $Y(s)$?

- (a) $\frac{2}{s(s^2 + 16)}$
- (b) $\frac{-2}{s^2 + 16}$
- (c) $\frac{2}{s^2 + 16}$
- (d) $\frac{2s}{s^2 + 16}$

✓

Feedback: Transform the equation with $\mathcal{L}\{\ddot{y}\} = s^2Y - sy(0) - \dot{y}(0)$ and solve for Y .

- (a) The equation has no input, and the initial condition enters as a constant, not as a step $2/s$.
- (b) The initial-condition term $-\dot{y}(0)$ moves to the right-hand side as $+2$.
- (c) Correct. $s^2Y - 2 + 16Y = 0$ gives $Y = 2/(s^2 + 16)$, so $y = \frac{1}{2} \sin 4t$.
- (d) In the derivative rule s multiplies $y(0)$, not $\dot{y}(0)$.

15. Solve $\ddot{y} + 5\dot{y} + 6y = 6u(t)$ with $u(t) = H(t)$, $y(0^-) = 2$, $\dot{y}(0^-) = -1$. Give $y(1)$.

Answer: 1.221

Feedback: Transforming gives $(s^2 + 5s + 6)Y = 6/s + 2s + 10 - 1$, so $Y = \frac{2s^2 + 9s + 6}{s(s+2)(s+3)} = \frac{1}{s} + \frac{2}{s+2} - \frac{1}{s+3}$. Then $y(t) = 1 + 2e^{-2t} - e^{-3t}$ and $y(1) = 1.221$.

16. Solve $\ddot{y} + 4\dot{y} + 4y = 2u(t)$ with $u(t) = H(t)$, $y(0^-) = 0.5$, $\dot{y}(0^-) = 0$. For $t \geq 0$, $y(t) =$

(a) $0.5 + 0.5e^{-2t} + te^{-2t}$

(b) $0.5 - 0.5e^{-2t} - te^{-2t}$

(c) 0.5

✓

(d) $0.5e^{-2t} + te^{-2t}$

Feedback: Transform the equation including both the input and the initial conditions, then solve for $Y(s)$.

- (a) This adds the zero-input response to the final value 0.5 but leaves out the transient of the zero-state response.
- (b) This is the zero-state response only, which leaves out the initial conditions.
- (c) Correct. $(s+2)^2 Y = 2/s + 0.5s + 2 = 0.5(s+2)^2/s$, so $Y = 0.5/s$ and the system starts at its equilibrium.
- (d) This is the zero-input response only, which leaves out the input.
17. A current source $u(t) = tH(t)$ A drives a capacitor $C = 0.25$ F in parallel with an inductor $L = 1$ H. Both start with zero energy. Find the maximum value of the voltage $y(t)$ across them for $t \geq 0$, in volts.

Answer: 2.000 V

Feedback: The parallel impedance is $Z(s) = \frac{sL}{s^2LC+1} = \frac{4s}{s^2+4}$, so $Y(s) = \frac{1}{s^2} \cdot \frac{4s}{s^2+4} = \frac{1}{s} - \frac{s}{s^2+4}$. Then $y(t) = 1 - \cos 2t$, whose maximum is 2.000 V.

Initial and final value theorems

18. A system has $G(s) = \frac{120}{s^2 + 6s}$. Use the final value theorem to find $\lim_{t \rightarrow \infty} h(t)$ for the impulse response $h(t)$.

Answer: 20.00

Feedback: $sG(s) = \frac{120}{s+6}$ has its only pole at -6 , so the final value theorem applies. The limit is $sG(s)$ at $s = 0$, which is $120/6 = 20.00$.

19. The system $G(s) = \frac{s+1}{s+10}$ is driven by a unit step $u(t) = H(t)$ from rest. What are $y(0^+)$ and $\lim_{t \rightarrow \infty} y(t)$?

(a) $y(0^+) = 0$, $y(\infty) = 0.1$

(b) $y(0^+) = 0.1$, $y(\infty) = 1$

(c) $y(0^+) = 1$, $y(\infty) = 0$

(d) $y(0^+) = 1$, $y(\infty) = 0.1$

✓

Feedback: With $Y(s) = G(s)/s$, both theorems use $sY(s) = G(s)$: the initial value is its limit as $s \rightarrow \infty$ and the final value is its value at $s = 0$.

- (a) $G(s)$ has equal numerator and denominator degree, so the step response jumps at $t = 0$ and $y(0^+) = 1$.
 - (b) This swaps the two values: $G(0) = 0.1$ is the final value and $\lim_{s \rightarrow \infty} G = 1$ is the initial value.
 - (c) The final value is $G(0) = 1/10$, which is 0 only if G has a zero at the origin.
 - (d) Correct. $\lim_{s \rightarrow \infty} G(s) = 1$ and $G(0) = 1/10$.
20. An RC lowpass with $G(s) = \frac{1}{2s+1}$ (time in seconds) is driven by the ramp $u(t) = tH(t)$ V, starting from rest. Find $\lim_{t \rightarrow \infty} (u(t) - y(t))$ in volts.

Answer: 2.000 V

Feedback: The difference has transform $E(s) = \frac{1}{s^2} \left(1 - \frac{1}{2s+1}\right) = \frac{2}{s(2s+1)}$. The only pole of $sE(s)$ is at $-1/2$, so the final value theorem applies and gives $\lim_{s \rightarrow 0} sE(s) = 2.000$ V.

21. For which one of these transfer functions does the final value theorem correctly give $\lim_{t \rightarrow \infty} h(t)$ for the impulse response?
- (a) $\frac{12}{s-4}$
 - (b) $\frac{12}{s^2+4}$
 - (c) $\frac{12}{s^4+4}$
 - (d) $\frac{12}{s(s+4)}$ ✓

Feedback: The final value theorem gives the correct limit only when every pole of $sG(s)$ is in the open left half plane.

- (a) The pole at $+4$ gives $h(t) = 12e^{4t}H(t)$, which grows without bound.
- (b) The poles at $\pm 2j$ give $h(t) = 6\sin(2t)H(t)$, which oscillates and has no limit.
- (c) $s^4 + 4$ has poles at $1 \pm j$ in the right half plane, so $h(t)$ grows without bound.
- (d) Correct. $sG(s) = 12/(s+4)$ has its pole at -4 , and the limit is $12/4 = 3$.

2. System analysis with the Laplace transform (week 9)

Transfer functions, poles and stability

22. Classify $G(s) = \frac{12}{s^2+4}$.

- (a) BIBO stable but not exponentially stable
- (b) Marginally stable and not BIBO stable ✓
- (c) Unstable
- (d) Exponentially stable

Feedback: Find the poles and check where they lie relative to the imaginary axis.

- (a) BIBO stability needs $\int_0^\infty |h(t)| dt$ to be finite, and $|6 \sin 2t|$ has an infinite integral.
 - (b) Correct. The poles $\pm 2j$ are simple and on the imaginary axis, and $h(t) = 6 \sin(2t)H(t)$ is bounded but not absolutely integrable.
 - (c) Simple poles on the imaginary axis give a bounded $h(t)$, which is marginal stability, not instability.
 - (d) Exponential stability needs all poles in the open left half plane, but the poles $\pm 2j$ are on the imaginary axis.
23. A transfer function $G(s)$ has poles at 0 and $-0.2 \pm j$, with no pole-zero cancellation. Which description is correct?
- (a) Marginally stable, but not BIBO stable ✓
 - (b) Marginally stable and BIBO stable
 - (c) Exponentially stable
 - (d) Unstable

Feedback: Check each pole: the pair $-0.2 \pm j$ is in the open left half plane, and the pole at 0 is a simple pole on the imaginary axis.

- (a) Correct. All poles are in the closed left half plane with the imaginary-axis pole simple, and the constant term in $h(t)$ rules out BIBO stability.
 - (b) The pole at 0 adds a nonzero constant to $h(t)$, so $\int_0^\infty |h| dt$ is infinite.
 - (c) The pole at $s = 0$ is not in the open left half plane, so the system is not exponentially stable.
 - (d) A simple pole on the imaginary axis gives a bounded $h(t)$, which is marginal stability, not instability.
24. Which transfer function is marginally stable?

- (a) $\frac{s+2}{(s^2+4)(s+1)}$ ✓
- (b) $\frac{s-2}{(s+1)(s+3)}$
- (c) $\frac{s+2}{s^2(s+3)}$
- (d) $\frac{s+2}{(s^2+4)^2}$

Feedback: Marginal stability means all poles are in the closed left half plane, with every imaginary-axis pole simple and at least one such pole present.

- (a) Correct. The poles $\pm 2j$ are simple and on the imaginary axis, and the pole -1 is in the open left half plane.
- (b) Both poles are in the open left half plane, so this is exponentially stable, and the zero at $+2$ does not affect stability.

- (c) The pole at $s = 0$ is repeated, so $h(t)$ grows like t and the system is unstable.
- (d) The poles $\pm 2j$ are repeated, so $h(t)$ grows like $t \sin 2t$ and the system is unstable.

25. Classify $G(s) = \frac{12}{s^4 + 4}$. (Hint: $s^4 + 4 = (s^2 + 2s + 2)(s^2 - 2s + 2)$.)

- (a) Exponentially stable, since all denominator coefficients are positive
- (b) BIBO stable but not exponentially stable
- (c) Marginally stable, with poles on the imaginary axis like $s^2 + 4$
- (d) Unstable ✓

Feedback: Use the hint to find the four poles from the two quadratic factors.

- (a) The s^3 , s^2 and s coefficients are zero, and the factor $s^2 - 2s + 2$ has poles at $1 \pm j$ in the right half plane.
- (b) With right half plane poles and no cancellation, $h(t)$ grows without bound and the system is not BIBO stable.
- (c) The poles are $-1 \pm j$ and $1 \pm j$, none of which is on the imaginary axis.
- (d) Correct. The factor $s^2 - 2s + 2$ gives poles at $1 \pm j$ in the right half plane.

26. For $G(s) = \frac{1}{s^2 + 9}$ with zero initial conditions, which bounded input produces an unbounded output?

- (a) $\cos(3t)H(t)$ ✓
- (b) $H(t)$
- (c) $\cos(9t)H(t)$
- (d) $e^{-3t}H(t)$

Feedback: The poles of G are at $\pm 3j$, and an input whose transform has poles at the same place produces repeated poles in $Y(s)$.

- (a) Correct. $Y = \frac{s}{(s^2+9)^2}$ gives $y(t) = \frac{t}{6} \sin 3t$, which grows without bound.
- (b) $Y = \frac{1}{s(s^2+9)}$ gives $y(t) = \frac{1}{9}(1 - \cos 3t)$, which is bounded.
- (c) The poles of G are at $\pm 3j$ because $\omega^2 = 9$, so an input at frequency 9 gives a bounded response.
- (d) This input decays and its pole at -3 does not match a pole of G , so the response is bounded.

Partial fractions and inverse transforms

27. Find the impulse response of $G(s) = \frac{s+2}{s(s+1)}$ at $t = 1$ s.

Answer: 1.632

Feedback: Write $G(s) = \frac{A}{s} + \frac{B}{s+1}$. Cover-up gives $A = 2/1 = 2$ and $B = (-1+2)/(-1) = -1$, so $h(t) = (2 - e^{-t})H(t)$ and $h(1) = 2 - e^{-1} = 1.632$.

28. Write $Y(s) = \frac{6}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$. Find A .

Answer: -0.6667

Feedback: Cover-up gives $B = 6/3 = 2$ and $C = 6/9 = 2/3$, but it does not give A directly. Matching the s^2 coefficient in $6 = As(s+3) + B(s+3) + Cs^2$ gives $A + C = 0$, so $A = -2/3 = -0.6667$.

29. What is the inverse Laplace transform of $Y(s) = \frac{2s+5}{s^2+2s+1}$?

(a) $2\delta(t) + 3te^{-t}H(t)$

(b) $(2+3t)e^{-t}H(t)$

✓

(c) $(3+2t)e^{-t}H(t)$

(d) $(2+5t)e^{-t}H(t)$

Feedback: The denominator is $(s+1)^2$, so write $Y = \frac{A}{s+1} + \frac{B}{(s+1)^2}$ and use $\frac{1}{(s+1)^2} \leftrightarrow te^{-t}H(t)$.

(a) $Y(s)$ is strictly proper, so its inverse has no impulse.

(b) Correct. $B = 2(-1) + 5 = 3$ and matching the s coefficient gives $A = 2$.

(c) This swaps A and B , putting the $1/(s+1)^2$ coefficient on e^{-t} .

(d) This takes B as the numerator at $s = 0$, but cover-up evaluates it at $s = -1$, giving $B = 3$.

30. The inverse transform of $Y(s) = \frac{3s+1}{(s+1)(s^2+2s+5)}$ has the form

$$y(t) = Ae^{-t} + e^{-t}(B \cos 2t + C \sin 2t), \quad t \geq 0.$$

Find C .

Answer: 1.5

Feedback: Cover-up gives $A = -2/4 = -0.5$, and matching coefficients in $3s+1 = A(s^2+2s+5) + (Ds+E)(s+1)$ gives $D = 0.5$ and $E = 3.5$. Completing the square, $\frac{0.5s+3.5}{(s+1)^2+4} = 0.5\frac{s+1}{(s+1)^2+4} + 1.5\frac{2}{(s+1)^2+4}$, so $C = 1.5$.

31. $G(s) = \frac{s+3}{(s+1)(s+3.1)}$ has a zero very close to a pole. Find the coefficient of $e^{-3.1t}$ in the impulse response $g(t)$.

Answer: 0.04762

Feedback: Cover-up at $s = -3.1$ gives the coefficient $\frac{-3.1+3}{-3.1+1} = \frac{-0.1}{-2.1} = 0.04762$. The zero close to the pole makes this residue small.

32. Find $y(3)$ where $Y(s) = \frac{4e^{-s}}{(s+1)(s+3)}$.

Answer: 0.2657

Feedback: Expand $\frac{4}{(s+1)(s+3)} = \frac{2}{s+1} - \frac{2}{s+3}$ and apply the delay of 1 from e^{-s} , so $y(t) = 2(e^{-(t-1)} - e^{-3(t-1)})H(t-1)$. Then $y(3) = 2(e^{-2} - e^{-6}) = 0.2657$.

33. $G(s) = \frac{s+4}{s^2+3s+2}$ is driven by $u(t) = e^{-2t}H(t)$ with zero initial conditions. Find $y(1)$.

Answer: 0.4270

Feedback: The input pole at -2 matches a system pole, so $Y = \frac{s+4}{(s+1)(s+2)^2} = \frac{3}{s+1} - \frac{3}{s+2} - \frac{2}{(s+2)^2}$. Then $y(t) = 3e^{-t} - 3e^{-2t} - 2te^{-2t}$ and $y(1) = 0.4270$.

34. An RC lowpass with $G(s) = \frac{1}{s+1}$ starts from rest and is driven by $u(t) = (1 - e^{-3t})H(t)$ V. Find $y(1)$ in volts.

Answer: 0.4731 V

Feedback: $U(s) = \frac{1}{s} - \frac{1}{s+3} = \frac{3}{s(s+3)}$, so $Y(s) = \frac{3}{s(s+1)(s+3)} = \frac{1}{s} - \frac{3/2}{s+1} + \frac{1/2}{s+3}$ by cover-up. Then $y(1) = 1 - 1.5e^{-1} + 0.5e^{-3} = 0.4731$ V.

35. Two identical RC sections, each with transfer function $\frac{2}{s+2}$, are connected through an ideal unity-gain buffer, so the overall transfer function is $\frac{4}{(s+2)^2}$. A unit step is applied from rest. Find the output at $t = 1$ s.

Answer: 0.5940

Feedback: $Y(s) = \frac{4}{s(s+2)^2} = \frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+2)^2}$, so $y(t) = 1 - e^{-2t} - 2te^{-2t}$. Then $y(1) = 1 - 3e^{-2} = 0.5940$.

36. A series RLC circuit ($L = 1$ H, $C = 0.25$ F, $R = 2$ Ω , output across C , starting from rest) has $G(s) = \frac{4}{s^2+2s+4}$. Which input $u(t)$ makes the output exactly $y(t) = (1 - e^{-4t})H(t)$?

(a) $u(t) = \delta(t) + (1 - 3e^{-4t})H(t)$

✓

(b) $u(t) = (1 - 3e^{-4t})H(t)$

(c) No such input exists, because the circuit is underdamped

(d) $u(t) = (1 - e^{-4t})H(t)$

Feedback: Solve for the input as $U(s) = Y(s)/G(s)$, then invert it.

(a) Correct. $U(s) = \frac{s^2+2s+4}{s(s+4)} = 1 + \frac{1}{s} - \frac{3}{s+4}$, and the constant 1 is an impulse.

(b) $U(s)$ is improper, and this leaves out the constant term from the division, which is the impulse $\delta(t)$.

(c) The factor $s^2 + 2s + 4$ appears in the numerator of $U(s) = Y(s)/G(s)$, so the input cancels the complex poles and $y(t)$ need not oscillate.

(d) The output equals the input only when $G(s) = 1$, which is not the case here.

Zeros

37. $G(s)$ has poles at -2 and -5 , a single zero at -1 , and DC gain 3. The input is $u(t) = e^{-t}H(t)$, starting from rest. What is $y(t)$?

(a) $(e^{-2t} - e^{-5t})H(t)$

(b) $y(t) = 0$, because the zero at -1 blocks e^{-t}

(c) $10(e^{-2t} - e^{-5t})H(t)$

✓

(d) $y(t)$ contains a nonzero e^{-t} term as well as e^{-2t} and e^{-5t} terms

Feedback: Write $G(s) = \frac{k(s+1)}{(s+2)(s+5)}$ and fix k from $G(0) = k/10 = 3$, then form $Y(s) = G(s)U(s)$ with $U(s) = 1/(s+1)$.

(a) This takes $k = 3$, but the DC gain is $G(0) = k/10$, so $k = 30$.(b) The zero removes the e^{-t} term from the output, but the system modes e^{-2t} and e^{-5t} are still excited from rest.(c) Correct. $k = 30$, the zero cancels the input pole, and $Y(s) = \frac{30}{(s+2)(s+5)} = 10\left(\frac{1}{s+2} - \frac{1}{s+5}\right)$.(d) The zero at -1 cancels the pole of $U(s)$ at -1 , so $Y(s)$ has no pole at -1 and $y(t)$ has no e^{-t} term.

38. $G(s) = 1 + \frac{1}{s} + 0.25s$. Which describes the straight-line Bode magnitude?

(a) 0 dB/decade below 2 rad/s, then +40 dB/decade

(b) -20 dB/decade below 2 rad/s, then +20 dB/decade

✓

(c) -20 dB/decade below 2 rad/s, then +40 dB/decade

(d) -20 dB/decade below 2 rad/s, then 0 dB/decade

Feedback: Put $G(s)$ over a common denominator and factor it to find the poles and zeros.

(a) The term $1/s$ is a pole at the origin, so the slope below 2 rad/s is -20 dB/decade.(b) Correct. $G(s) = (s+2)^2/(4s)$ has a pole at 0 and a double zero at -2 , so the slope goes from -20 to $-20 + 40 = +20$ dB/decade.

(c) The pole at the origin still contributes -20 dB/decade above 2 rad/s, so the slope there is +20, not +40.

(d) The numerator $(s+2)^2$ is a double zero, so the slope rises by 40 dB/decade at 2 rad/s, not 20.

39. A filter $G(s) = K \frac{s+z}{s+5}$ has DC gain 1 and must have a high-frequency gain of -30 dB. Where must the zero be? Give z in rad/s (the zero is at $s = -z$).

Answer: 158.1 rad/s

Feedback: The DC gain is $G(0) = Kz/5 = 1$ and the high-frequency gain is $K = 10^{-30/20} = 0.03162$. Then $z = 5/K = 158.1$ rad/s.

40. A 50 Hz hum notch filter is $G(s) = \frac{s^2 + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ with $\omega_n = 2\pi \cdot 50$ rad/s and $\zeta = 0.5$. Find $|G(j\omega)|$ at 25 Hz.

Answer: 0.8321

Feedback: At 25 Hz, $\omega = \omega_n/2$, so $G(j\omega) = \frac{0.75\omega_n^2}{0.75\omega_n^2 + j0.5\omega_n^2}$. Then $|G| = 0.75/\sqrt{0.75^2 + 0.5^2} = 0.75/\sqrt{0.8125} = 0.8321$.

3. Filter design (week 10)

Filter specifications and first- and second-order filters

41. Why can an ideal lowpass filter $G(j\omega) = p_{2\omega_c}(\omega)$ not be built as a real-time circuit?

- (a) Its gain is infinite at ω_c
- (b) Its phase is not linear
- (c) Its impulse response $\sin(\omega_c t)/(\pi t)$ is nonzero for $t < 0$, so it is noncausal ✓
- (d) It has a pole in the right half plane

Feedback: A real-time system must be causal, so its impulse response $h(t)$ must be zero for $t < 0$. Find $h(t)$ as the inverse Fourier transform of $G(j\omega)$.

- (a) The gain is 1 in the passband and 0 outside it, so it is bounded.
- (b) The ideal filter has zero phase at every frequency, which is linear.
- (c) Correct. The inverse Fourier transform of the pulse is $\sin(\omega_c t)/(\pi t)$, which extends to $t \rightarrow -\infty$.
- (d) The ideal filter has no rational transfer function, so it has no poles at all.

42. At the -3 dB frequency of a first-order RC lowpass, compared with DC, the output

- (a) amplitude is $1/2$ and power is $1/4$
- (b) amplitude is $1/\sqrt{2}$ and power is $1/2$ ✓
- (c) amplitude and power are both 0.7
- (d) amplitude is $1/2$ and power is $1/\sqrt{2}$

Feedback: The -3 dB point is where $20 \log_{10} |G| = -3$, and power is proportional to the square of the amplitude.

- (a) An amplitude ratio of $1/2$ is -6 dB, not -3 dB.
- (b) Correct. $20 \log_{10}(1/\sqrt{2}) = -3.01$ dB, and the power ratio is $|G|^2 = 1/2$.
- (c) The amplitude ratio is about 0.7 , but the power ratio is its square, $1/2$.
- (d) This swaps the amplitude and power ratios, but the power ratio must be the square of the amplitude ratio.

43. Which transfer function is a bandpass filter?

- (a) $\frac{10s}{s^2 + 10s + 100}$ ✓
- (b) $\frac{100}{s^2 + 10s + 100}$
- (c) $\frac{s^2 + 100}{s^2 + 10s + 100}$
- (d) $\frac{s^2}{s^2 + 10s + 100}$

Feedback: Find the gain at $\omega = 0$ and as $\omega \rightarrow \infty$. A bandpass filter has zero gain at both ends.

- (a) Correct. The zero at $s = 0$ gives $G(0) = 0$, the gain goes to 0 at high frequency because G is strictly proper, and $|G(j10)| = 1$.
- (b) The gain is 1 at DC and tends to 0 at high frequency, so this is a lowpass filter.
- (c) The zeros at $\pm 10j$ give zero gain at 10 rad/s and unit gain at both ends, so this is a bandstop filter.
- (d) The gain is 0 at DC and tends to 1 at high frequency, so this is a highpass filter.

44. Find the -3 dB frequency of the second-order lowpass $G(s) = \frac{400}{s^2 + 20s + 400}$, in rad/s.

Answer: 25.44 rad/s

Feedback: Here $\omega_n = 20$ and $\zeta = 0.5$, so with $x = (\omega/\omega_n)^2$, $|G|^2 = 1/((1-x)^2 + x)$. Setting this to $1/2$ gives $x^2 - x - 1 = 0$, so $x = 1.618$ and $\omega = 20\sqrt{1.618} = 25.44$ rad/s.

45. A first-order highpass $G(s) = \frac{s}{s + \omega_c}$ with $\omega_c = 2\pi \cdot 100$ rad/s is used to suppress a 50 Hz hum. By how many dB is the 50 Hz hum attenuated? (Give a positive number.)

Answer: 6.990 dB

Feedback: With $f/f_c = 50/100 = 0.5$, $|G| = 0.5/\sqrt{1+0.5^2} = 0.4472$. The attenuation is $-20 \log_{10}(0.4472) = 6.990$ dB.

46. A first-order lowpass $G(s) = \frac{k}{s + a}$ with $k, a > 0$ must have DC gain +12 dB and a gain 3 dB below the DC gain at $f = 1$ kHz. Find k (units rad/s).

Answer: 2.501×10^4 rad/s

Feedback: The gain is 3 dB below the DC gain at $\omega = a$, so $a = 2\pi \times 1000 = 6283$ rad/s. The DC gain is $k/a = 10^{12/20} = 3.981$, so $k = 3.981 \times 6283 = 2.501 \times 10^4$ rad/s.

47. Which transfer function is an exponentially stable allpass filter ($|G(j\omega)| = 1$ for all ω)?

- (a) $\frac{s}{s+3}$
- (b) $\frac{3}{s+3}$
- (c) $\frac{s+3}{s-3}$
- (d) $\frac{s-3}{s+3}$

✓

Feedback: A first-order allpass has its zero at the mirror image of its pole in the imaginary axis, and exponential stability needs the pole in the open left half plane.

- (a) This is a first-order highpass, with gain 0 at DC rising to 1 at high frequency.
- (b) This is a first-order lowpass, with gain 1 at DC falling to 0 at high frequency.
- (c) This has unit gain, but its pole at +3 makes it unstable.
- (d) Correct. $|j\omega - 3| = |j\omega + 3|$ gives unit gain, and the pole at -3 is in the open left half plane.

48. The allpass filter $G(s) = \frac{10-s}{s+10}$ is driven by the input $u(t) = \cos(10t)$, applied for all time. The output is

- (a) $-\cos(10t)$
- (b) $\cos(10t)$
- (c) $\sin(10t)$
- (d) $-\sin(10t)$

✓

Feedback: Evaluate $G(j10)$. The output is then $|G(j10)| \cos(10t + \angle G(j10))$.

- (a) This uses the high-frequency phase -180° , but at 10 rad/s the phase is -90° .
 - (b) The gain is 1, but the phase at 10 rad/s is -90° , not 0.
 - (c) Correct. $G(j10) = \frac{10-10j}{10+10j} = -j$, so $y(t) = \cos(10t - 90^\circ) = \sin(10t)$.
 - (d) The phase is -90° , and $\cos(10t - 90^\circ) = +\sin(10t)$.
49. The system $y(t) = u(t - 2)$ has transfer function $G(s) = e^{-2s}$. Which describes its frequency response?

- (a) It has a pole at $s = -2$, so $|G|$ falls at -20 dB/decade above 2 rad/s
- (b) $|G(j\omega)| = 1$ and $\angle G(j\omega) = -2\omega$ rad: an allpass
- (c) $|G(j\omega)| = e^{-2\omega}$: a lowpass
- (d) $|G(j\omega)| = 1$ and $\angle G(j\omega) = -2$ rad at all ω

✓

Feedback: Substitute $s = j\omega$ to get $G(j\omega) = e^{-j2\omega}$, then read off the magnitude and phase.

- (a) e^{-2s} is not a rational function, so it has no poles.
- (b) Correct. $|e^{-j2\omega}| = 1$ at every frequency, and the phase -2ω is linear in ω .
- (c) This substitutes $s = \omega$ instead of $s = j\omega$.
- (d) The phase of $e^{-j2\omega}$ is -2ω , which grows with frequency.

Lowpass to highpass, bandpass and bandstop

50. The normalized Butterworth prototype $G_{LP}(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$ has cutoff 1 rad/s. Replacing s by $50/s$ gives

- (a) $\frac{s^2}{s^2 + 1.414s + 2500}$
- (b) $\frac{s^2}{s^2 + 70.71s + 2500}$
- (c) $\frac{s^2}{s^2 + 70.71s + 50}$
- (d) $\frac{2500}{s^2 + 70.71s + 2500}$

✓

Feedback: Substitute $50/s$ for s in the prototype and multiply the numerator and denominator by s^2 .

- (a) The middle coefficient is also scaled by 50, giving $50\sqrt{2} = 70.71$.
- (b) Correct. $\frac{1}{2500/s^2 + 50\sqrt{2}/s + 1} = \frac{s^2}{s^2 + 50\sqrt{2}s + 2500}$, and $50\sqrt{2} = 70.71$.
- (c) The constant term is $50^2 = 2500$, not 50.
- (d) This is the lowpass with cutoff 50 rad/s, which comes from $s \rightarrow s/50$.
51. In a two-way loudspeaker crossover, the tweeter (a $4\ \Omega$ resistor) is fed through a series capacitor C , giving the highpass $G_H(s) = \frac{4Cs}{4Cs + 1}$. Find C in μF so the $-3\ \text{dB}$ frequency is 3 kHz.

Answer: 13.26 μF

Feedback: The $-3\ \text{dB}$ frequency of $G_H(s)$ is $\omega = 1/(4C)$. Setting $1/(4C) = 2\pi \times 3000$ gives $C = 1/(4 \cdot 2\pi \cdot 3000) = 13.26\ \mu\text{F}$.

52. A bandstop filter is built as the parallel sum of a first-order lowpass and a first-order highpass: $G(s) = \frac{10}{s + 10} + \frac{s}{s + 1000}$. Find $|G(j100)|$.

Answer: 0.01980

Feedback: Add the complex values, not the magnitudes: $\frac{10}{10 + 100j} = 0.0099 - 0.0990j$ and $\frac{100j}{1000 + 100j} = 0.0099 + 0.0990j$. The imaginary parts cancel, so $|G(j100)| = 0.01980$.

53. A first-order lowpass $G_L(s) = \frac{\omega_c}{s + \omega_c}$ and a first-order highpass $G_H(s) = \frac{s}{s + \omega_c}$ with the same cutoff are connected in parallel and summed, $G = G_L + G_H$. What is G ?

- (a) A bandstop filter with a notch at ω_c
- (b) A bandpass filter centered at ω_c
- (c) $|G(j\omega_c)| = \sqrt{2}$, with unit gain elsewhere
- (d) $G(s) = 1$: every frequency passes unchanged ✓

Feedback: Add the two transfer functions over their common denominator $s + \omega_c$.

- (a) A parallel bandstop needs the lowpass cutoff below the highpass cutoff, and here the two cutoffs are equal.
- (b) A bandpass comes from a cascade of a lowpass and a highpass, not a parallel sum.
- (c) This adds the magnitudes $1/\sqrt{2} + 1/\sqrt{2}$, but the complex values $G_L(j\omega_c) = (1 - j)/2$ and $G_H(j\omega_c) = (1 + j)/2$ sum to 1.
- (d) Correct. $G(s) = \frac{\omega_c + s}{s + \omega_c} = 1$.

Butterworth filters and order selection

54. A second-order lowpass with no zeros has poles at $-10^4 \pm j10^4\ \text{rad/s}$ and unit DC gain. It is a Butterworth filter. Find its $-3\ \text{dB}$ cutoff ω_c in rad/s.

Answer: $1.414 \times 10^4\ \text{rad/s}$

Feedback: The poles have $\omega_n = |-10^4 + j10^4| = \sqrt{2} \times 10^4$ and $\zeta = 10^4/\omega_n = 1/\sqrt{2}$, which is the second-order Butterworth form. For a Butterworth filter the $-3\ \text{dB}$ cutoff equals ω_n , so $\omega_c = 1.414 \times 10^4\ \text{rad/s}$.

55. A second-order Butterworth lowpass with $f_c = 100$ Hz has transfer function $\frac{\omega_c^2}{s^2 + a s + \omega_c^2}$. Find a in rad/s.

Answer: 888.6 rad/s

Feedback: The second-order Butterworth denominator is $s^2 + \sqrt{2}\omega_c s + \omega_c^2$ with $\omega_c = 2\pi \cdot 100 = 628.3$ rad/s. So $a = \sqrt{2} \times 628.3 = 888.6$ rad/s.

56. A fifth-order Butterworth lowpass has $\omega_c = 1000$ rad/s. Find the real part of the pole pair closest to the imaginary axis.

Answer: -309.0 rad/s

Feedback: The poles lie on the circle of radius 1000 in the left half plane, 36° apart, at angles $108^\circ, 144^\circ, 180^\circ, 216^\circ, 252^\circ$. The pair closest to the imaginary axis is at 108° and 252° , with real part $1000 \cos 108^\circ = -309.0$ rad/s.

57. A third-order Butterworth highpass filter with cutoff frequency 100 Hz is used to suppress a 50 Hz hum. By how many dB is the hum attenuated? (Give a positive number.)

Answer: 18.13 dB

Feedback: A third-order Butterworth highpass has $|G|^2 = 1/(1 + (f_c/f)^6)$, and here $f_c/f = 2$. The attenuation is $10 \log_{10}(1 + 2^6) = 10 \log_{10} 65 = 18.13$ dB.

58. Two identical second-order Butterworth lowpass sections, each with cutoff ω_c , are cascaded. Which statement about the cascade is correct?

- (a) The gain at ω_c is -6.02 dB, and the cascade is a fourth-order Butterworth filter with cutoff ω_c
- (b) The gain at ω_c is -3.01 dB, and the cascade is a fourth-order Butterworth filter with cutoff ω_c
- (c) The gain at ω_c is -12.04 dB, and the cascade is not a fourth-order Butterworth filter
- (d) The gain at ω_c is -6.02 dB, and the cascade is not a fourth-order Butterworth filter ✓

Feedback: In a cascade the gains multiply, and a Butterworth filter of order n has n distinct poles equally spaced on a circle.

- (a) The gain is right, but a fourth-order Butterworth is -3 dB at its cutoff and has four distinct poles, while the cascade has two double poles.
- (b) Each section is -3.01 dB at ω_c , so the cascade is $2 \times (-3.01) = -6.02$ dB there.
- (c) Each section loses 3.01 dB at ω_c , not 6.02 dB, so the cascade loses 6.02 dB.
- (d) Correct. The gain is $(1/\sqrt{2})^2 = 1/2$, or -6.02 dB, and the cascade has double poles at 135° and 225° instead of the four distinct poles at $112.5^\circ, 157.5^\circ, 202.5^\circ, 247.5^\circ$ of a fourth-order Butterworth.

59. A second-order Butterworth highpass has cutoff 6 kHz, so $|G(j\omega)|^2 = 1/(1 + (\omega_c/\omega)^4)$. At what frequency, in Hz, is its gain -30 dB?

Answer: 1067 Hz

Feedback: Set $10 \log_{10} (1 + (f_c/f)^4) = 30$, so $(f_c/f)^4 = 10^3 - 1 = 999$. Then $f = 6000/999^{1/4} = 1067$ Hz.

60. A fourth-order Butterworth lowpass must lose at most 1 dB at 1 kHz. What is the smallest cutoff frequency f_c (the -3 dB frequency), in kHz?

Answer: 1.184 kHz

Feedback: The loss at 1 kHz is $10 \log_{10} (1 + (f/f_c)^8) \leq 1$ dB, so $(f/f_c)^8 \leq 10^{0.1} - 1 = 0.2589$ and $f/f_c \leq 0.8446$. Then $f_c \geq 1/0.8446 = 1.184$ kHz.

61. A Butterworth lowpass filter with cutoff frequency 2 kHz must attenuate a 6 kHz signal by at least 35 dB. Enter the smallest order n that meets this.

Answer: 4

Feedback: The frequency ratio is $6/2 = 3$, and the attenuation is $10 \log_{10}(1 + 3^{2n})$. This gives 28.6 dB for $n = 3$ and 38.2 dB for $n = 4$, so the smallest order is 4.

4. Modulation (week 11)

Amplitude modulation and demodulation

62. An antenna's length must be comparable to the wavelength $\lambda = c/f$ of the signal it transmits, with $c = 3 \times 10^8$ m/s. Find the wavelength of a 3 kHz telephone audio signal sent directly as an electromagnetic wave, in km.

Answer: 100 km

Feedback: Baseband audio has a very long wavelength, which is why it is modulated onto a carrier before transmission. Here $\lambda = c/f = 3 \times 10^8 / (3 \times 10^3) = 10^5$ m = 100 km.

63. The message $u(t) = 2 \cos(1000t) + \cos(2000t)$ is transmitted as $y(t) = u(t) \cos(10\,000t)$. Which list gives the positive frequencies (rad/s) in $y(t)$ with the amplitudes of the corresponding cosines?

- (a) 8000 (1), 9000 (2), 11 000 (2), 12 000 (1)
- (b) 1000 (2), 2000 (1), 10 000 (1)
- (c) 8000 (0.5), 9000 (1), 11 000 (1), 12 000 (0.5) ✓
- (d) 10 000 (1), 11 000 (1), 12 000 (0.5)

Feedback: Apply $\cos a \cos b = \frac{1}{2} [\cos(a - b) + \cos(a + b)]$ to each tone of $u(t)$.

- (a) This drops the factor $\frac{1}{2}$ in the product formula.
- (b) This is the spectrum of $u(t) + \cos(10\,000t)$, a sum rather than a product.
- (c) Correct. $2 \cos(1000t)$ gives cosines of amplitude 1 at 9000 and 11 000 rad/s, and $\cos(2000t)$ gives cosines of amplitude 0.5 at 8000 and 12 000 rad/s.
- (d) The product $u(t) \cos(10\,000t)$ has no carrier term, and it contains both sidebands.

64. A speech scrambler multiplies $u(t)$, bandlimited to 15 kHz, by $2 \cos(2\pi \cdot 15\,000t)$ and passes the product through an ideal lowpass filter with cutoff 15 kHz and gain 1. A component $3 \cos(2\pi \cdot 4000t)$ of $u(t)$ appears in the scrambled output as

- (a) $3 \cos(2\pi \cdot 19\,000t)$

- (b) $3 \cos(2\pi \cdot 4000 t)$
- (c) $3 \cos(2\pi \cdot 11\,000 t)$
- (d) $1.5 \cos(2\pi \cdot 11\,000 t)$

✓

Feedback: Multiply the tone by the carrier using $\cos a \cos b = \frac{1}{2} [\cos(a - b) + \cos(a + b)]$, then keep only the components below 15 kHz.

- (a) The 19 kHz term is above the 15 kHz cutoff, so the lowpass removes it.
- (b) The lowpass removes the 19 kHz term but does not undo the frequency shift, so the tone appears at $15 - 4 = 11$ kHz.
- (c) Correct. $3 \cos(\omega_m t) \cdot 2 \cos(\omega_0 t) = 3 \cos((\omega_0 - \omega_m)t) + 3 \cos((\omega_0 + \omega_m)t)$, and the lowpass keeps the 11 kHz term.
- (d) The carrier amplitude 2 cancels the factor $\frac{1}{2}$, so the amplitude stays 3.

Frequency division multiplexing

65. A radio band runs from 3.000 MHz to 3.400 MHz. Each telephone message has bandwidth 3.4 kHz and is sent by DSB-SC, with no guard bands and no overlap allowed. What is the largest number of channels that fit in the band?

Answer: 58

Feedback: Each DSB-SC channel occupies $2W = 6.8$ kHz, and the band is 400 kHz wide. Since $400/6.8 = 58.8$ and a partial channel does not fit, the largest number is 58.

66. Four messages, each bandlimited to 3 kHz, are sent by DSB-SC on carriers $f_1 < f_2 < f_3 < f_4$. The FDM spectrum must begin at 100 kHz, with 5 kHz guard bands between adjacent channels and no gaps otherwise. Find f_4 in kHz.

Answer: 136.0 kHz

Feedback: Each DSB-SC channel occupies 6 kHz centered on its carrier, so $f_1 = 103$ kHz and adjacent carriers are $6 + 5 = 11$ kHz apart. Then $f_4 = 103 + 3 \times 11 = 136.0$ kHz.

67. Two messages $u_1(t)$ and $u_2(t)$ are each bandlimited to 5000 rad/s. The signal $b(t) = u_1(t) + u_2(t) \cdot 2 \cos(10\,000t)$ is formed and then transmitted as $c(t) = b(t) \cdot 2 \cos(20\,000t)$. Which band of positive frequencies does $c(t)$ occupy?

- (a) 5000 to 35 000 rad/s
- (b) 10 000 to 30 000 rad/s
- (c) 0 to 35 000 rad/s
- (d) 15 000 to 25 000 rad/s

✓

Feedback: Find the band of $b(t)$ first, then shift it by the outer carrier frequency.

- (a) Correct. u_1 occupies 0 to 5000 rad/s and the modulated u_2 occupies 5000 to 15 000 rad/s, so $b(t)$ occupies 0 to 15 000 rad/s and $c(t)$ occupies $20\,000 \pm 15\,000$ rad/s.
- (b) This shifts the inner carrier frequency 10 000 rad/s instead of the full band of $b(t)$, which extends to 15 000 rad/s.
- (c) Multiplying by $2 \cos(20\,000t)$ moves the whole band of $b(t)$ up, so nothing remains near 0.

- (d) This is the band from u_1 alone, but $b(t)$ also contains the modulated u_2 out to 15 000 rad/s.

Frequency and phase modulation, instantaneous frequency

68. The signal $y(t) = \cos(t^2)$ is a sinusoid with variable frequency, $y(t) = \cos(\omega(t)t)$ with $\omega(t) = t$. Find its instantaneous frequency at $t = 5$ s, in rad/s.

Answer: 10 rad/s

Feedback: The instantaneous frequency is the derivative of the whole argument of the cosine, not $\omega(t)$. Here $\omega_i = \frac{d}{dt}(t^2) = 2t = 10$ rad/s at $t = 5$ s.

69. A PM transmitter sends $y(t) = \cos(\omega_c t + k u(t))$ with $k = 0.5$ rad/V. Over $0 \leq t \leq 1$ ms the message is a ramp, $u(t) = 4000 t$ V. By how many Hz is the instantaneous frequency shifted from the carrier during the ramp?

Answer: 318.3 Hz

Feedback: The instantaneous frequency is $\omega_c + k\dot{u}(t) = \omega_c + 0.5 \times 4000 = \omega_c + 2000$ rad/s. The shift is $2000/(2\pi) = 318.3$ Hz.

70. The message $u(t) = H(t)$ is sent once by FM, $y = \cos\left(\omega_c t + k \int_{-\infty}^t u\right)$, and once by PM, $y = \cos(\omega_c t + k u(t))$. Which description of the two signals for $t > 0$ is correct?

- (a) Both have frequency $\omega_c + k$ for $t > 0$.
- (b) Both multiply the carrier amplitude by $1 + k$ for $t > 0$.
- (c) FM: the frequency jumps to $\omega_c + k$ with continuous phase. PM: the phase jumps by k at $t = 0$ and the frequency is ω_c again for $t > 0$. ✓
- (d) FM: the phase jumps by k at $t = 0$. PM: the frequency jumps to $\omega_c + k$.

Feedback: Write the argument of each cosine for $t > 0$ and differentiate it to get the instantaneous frequency.

- (a) The PM argument $\omega_c t + k$ has derivative ω_c , so only the FM frequency changes.
 - (b) FM and PM change only the argument of the cosine, so the amplitude stays 1, and changing the amplitude is AM.
 - (c) Correct. The FM argument is $\omega_c t + kt$, with derivative $\omega_c + k$ and no jump at 0, and the PM argument is $\omega_c t + k$, with derivative ω_c .
 - (d) This swaps FM and PM.
71. A message $u(t)$ with spectrum zero for $|\omega| > W$ (and no DC component) is sent by FM, $y(t) = \cos\left(\omega_c t + k \int_{-\infty}^t u(\tau) d\tau\right)$, with $\omega_c \gg W$. For small but nonzero k , the spectrum of $y(t)$ consists of

- (a) lines at $\pm\omega_c$ plus sidebands in $\omega_c - W < |\omega| < \omega_c + W$, the same band as AM ✓
- (b) components spread over a band much wider than $2W$ around $\pm\omega_c$
- (c) lines at $\pm\omega_c$ only
- (d) the baseband spectrum $|\omega| < W$ plus lines at $\pm\omega_c$

Feedback: For small k , expand $\cos(\omega_c t + \theta) \approx \cos(\omega_c t) - \theta \sin(\omega_c t)$ with $\theta(t) = k \int_{-\infty}^t u(\tau) d\tau$.

- (a) Correct. $y(t) \approx \cos(\omega_c t) - k \sin(\omega_c t) \int u$ is a carrier plus AM of $\int u$, which has the same bandwidth W as u .
- (b) A band much wider than $2W$ appears only for large k .
- (c) This drops the first-order term $-k \sin(\omega_c t) \int u$, which is nonzero for $k \neq 0$.
- (d) The message multiplies the carrier, so no component remains at baseband.