

ELEC2302/9302 Signals and Systems — Quiz 3 formula sheet

Laplace transform

$$U(s) = \mathcal{L}\{u\}(s) = \int_{0^-}^{\infty} u(t) e^{-st} dt, \quad Y(s) = G(s) U(s), \quad G(s) = \mathcal{L}\{h\}(s).$$

$u(t)$	$U(s)$	$u(t)$	$U(s)$
$\delta(t)$	1	$e^{-at}H(t)$	$\frac{1}{s+a}$
$\delta(t-t_0), t_0 \geq 0$	e^{-st_0}	$t e^{-at}H(t)$	$\frac{1}{(s+a)^2}$
$H(t)$	$\frac{1}{s}$	$t^n e^{-at}H(t)$	$\frac{n!}{(s+a)^{n+1}}$
$t H(t)$	$\frac{1}{s^2}$	$e^{-at} \cos(\omega_0 t) H(t)$	$\frac{s+a}{(s+a)^2 + \omega_0^2}$
$t^n H(t)$	$\frac{n!}{s^{n+1}}$	$e^{-at} \sin(\omega_0 t) H(t)$	$\frac{\omega_0}{(s+a)^2 + \omega_0^2}$
$\cos(\omega_0 t) H(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\sin(\omega_0 t) H(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$

Properties of the Laplace transform

	$u(t)$	$U(s)$		$u(t)$	$U(s)$
linearity	$\alpha u_1(t) + \beta u_2(t)$	$\alpha U_1(s) + \beta U_2(s)$	differentiation	$\frac{d}{dt} u(t)$	$s U(s) - u(0^-)$
time shift	$u(t-t_0)H(t-t_0), t_0 \geq 0$	$e^{-st_0} U(s)$	second derivative	$\frac{d^2}{dt^2} u(t)$	$s^2 U(s) - s u(0^-) - \dot{u}(0^-)$
s-shift	$e^{-at} u(t)$	$U(s+a)$	integration	$\int_{0^-}^t u(\zeta) d\zeta$	$\frac{U(s)}{s}$
time scaling	$u(at), a > 0$	$\frac{1}{a} U\left(\frac{s}{a}\right)$	convolution	$(h * u)(t)$	$H(s) U(s)$
multiply by t	$t u(t)$	$-\frac{d}{ds} U(s)$	modulation	$u(t) \cos(\omega_0 t)$	$\frac{1}{2} U(s - j\omega_0) + \frac{1}{2} U(s + j\omega_0)$

Initial value theorem: $u(0^+) = \lim_{s \rightarrow \infty} s U(s)$,

Final value theorem: $\lim_{t \rightarrow \infty} u(t) = \lim_{s \rightarrow 0} s U(s)$.

Butterworth filters

$$|G(j\omega)|^2 = \frac{1}{1 + (\omega/\omega_c)^{2n}}, \quad n = 1: \frac{\omega_c}{s + \omega_c}, \quad n = 2: \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}, \quad n = 3: \frac{\omega_c^3}{(s + \omega_c)(s^2 + \omega_c s + \omega_c^2)}.$$

$$n \text{ even: } G(s) = \prod_{k=1}^{n/2} \frac{\omega_c^2}{s^2 + 2\zeta_k \omega_c s + \omega_c^2}, \quad n \text{ odd: } G(s) = \frac{\omega_c}{s + \omega_c} \prod_{k=1}^{(n-1)/2} \frac{\omega_c^2}{s^2 + 2\zeta_k \omega_c s + \omega_c^2}, \quad \zeta_k = \sin\left(\frac{(2k-1)\pi}{2n}\right).$$

Modulation

AM	$y(t) = u(t) \cos(\omega_c t), \quad Y(j\omega) = \frac{1}{2} U(j(\omega - \omega_c)) + \frac{1}{2} U(j(\omega + \omega_c))$
Demodulation	$z(t) = y(t) \cos(\omega_c t) = \frac{1}{2} u(t) + \frac{1}{2} u(t) \cos(2\omega_c t)$
FM	$y(t) = \cos\left(\omega_c t + k \int_{-\infty}^t u(\tau) d\tau\right)$
PM	$y(t) = \cos(\omega_c t + k u(t))$

First and second order systems

First order, $G(s) = \frac{K}{\tau s + 1}$: the step response reaches 63% of its final value at $t = \tau$, rise time (10% to 90%) $\approx 2.2\tau$, settling time (to within 5%) $\approx 3\tau$.

Second order, $G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, 0 < \zeta < 1$:

$$\text{settling time (5\%)} \approx \frac{3}{\zeta\omega_n}, \quad \text{resonant peak } \frac{|G(j\omega_r)|}{|K|} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \text{ at } \omega_r = \omega_n \sqrt{1-\zeta^2}, \quad \zeta < \frac{1}{\sqrt{2}}.$$

ζ	0.1	0.15	0.2	0.25	0.3	0.4	0.5	0.6	0.707
overshoot M_p	73%	62%	53%	44%	37%	25%	16%	9.5%	4.3%
resonant peak $ G(j\omega_r) / K $	5.03	3.37	2.55	2.07	1.75	1.36	1.15	1.04	1.00
resonant peak, dB above DC	14.0	10.6	8.1	6.3	4.8	2.7	1.2	0.4	0